Bézier Controllers and Observers for Takagi-Sugeno Models

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***Abstract*— This paper presents Bézier controller and observer designs for T-S models with n local models. These designs are based on the m-th multi-sum generalization of the Parallel Distributed Compensation (PDC) and non-PDC control laws, but where a Bézier interpolation of the gain matrices is considered: the gain matrices are weighted by multivariate Bernstein polynomials of the activation functions. This reduces the number of gains from n^m to $(m+n-1)!/m!(n-1)!$ without hindering the capabilities of the control law. For quadratic and nonquadratic Lyapunov functions, the resulting stabilization problems can be solved using simple LMIs. Some examples are provided to illustrate numerically the reduced conservatism of the optimization problems compared to the usual PDC and non-PDC approaches.**

I. Introduction

The Takagi-Sugeno (T-S) models [1] form a large class of nonlinear systems which have attracted a lot of attention in the literature due to their wide modeling capabilities, and their convenience in order to solve nonlinear control problems. Indeed, under the T-S framework, these control problems can generally be stated as Linear Matrix Inequalities (LMIs), which can be efficiently solved through convex optimization techniques [2], [3].

This paper is concerned with the stabilization problem for continuous-time T-S models of the following form

$$\begin{cases} \dot{x}(t) = & \sum_{i=1}^{n} h_i(\theta)\left[A_i x(t) + B_i u(t)\right] \\ y(t) = & \sum_{i=1}^{n} h_i(\theta) C_i x(t) \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$ the state, input and output vector respectively, and $\theta \in \Omega$ a scheduling vector assumed throughout the paper to be exactly measured in real-time. The $(h_i)_{1\le i\le n}$ are the activation functions depending on θ satisfying the convex sum properties, i.e.

$$h_i(\theta) \geq 0, \qquad \sum_{i=1}^{n} h_i(\theta) = 1 \tag{2}$$

For concision purposes, the activation functions $\mathbf{h}(\theta) \triangleq (h_1(\theta), \ldots, h_n(\theta))$ of the T-S model (1) are now denoted $\mathbf{h} \triangleq (h_1, \ldots, h_n)$.

The stabilization of such models is generally handled in the literature by considering the following Parallel Distributed Compensation (PDC) control law [4]

$$u(t) = \sum_{i=1}^{n} h_i K_i x(t) \tag{3}$$

which has an estimator counterpart [2], [3] consisting in the following T-S observer

$$\begin{cases} \dot{\hat{x}}(t) = & \sum_{i=1}^{n} h_i\left[A_i \hat{x}(t) + B_i u(t) + K_i[\hat{y}(t) - y(t)]\right] \\ \hat{y}(t) = & \sum_{i=1}^{n} h_i C_i \hat{x}(t) \end{cases} \tag{4}$$

The stabilization can also be handled with the following non-PDC control law [5]

$$u(t) = \left[\sum_{i=1}^{n} h_i K_i\right]\left[\sum_{j=1}^{n} h_j P_j\right]^{-1} x(t) \tag{5}$$

where $V(x) = x^\top\left[\sum_{j=1}^{n} h_j P_j\right]^{-1} x$ is used as a non-quadratic Lyapunov function (NQLF) for the closed-loop system. Similarly, the following non-PDC observer [3] can be designed

$$\begin{cases} \dot{\hat{x}}(t) = & \sum_{i=1}^{n} h_i\Big[A_i \hat{x}(t) + B_i u(t) \\ & +[\sum_{j=1}^{n} h_j P_j]^{-1} K_i[\hat{y}(t) - y(t)]\Big] \\ \hat{y}(t) = & \sum_{i=1}^{n} h_i C_i \hat{x}(t) \end{cases} \tag{6}$$

using $V(e) = e^\top\left[\sum_{j=1}^{n} h_j P_j\right] e$ as a NQLF to study the convergence of the observation error $e = \hat{x} - x$.

Each of these solutions involves the computation of a set of gain matrices $(K_i)_{1\le i\le n}$, resulting in an optimization problem with more degrees of freedom compared to simpler designs where a single constant gain matrix K is used. However, when this approach is still too conservative, meaning when the set of gain matrices is not sufficient to ensure the stability of the closed-loop system or of the observation error, then, the "natural generalization" considered by the literature consists in the introduction of multiple convex sums [6], [7], e.g. the generalization of the PDC control law (3) uses the multi-sum

$$\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} h_{i_1} \ldots h_{i_m} K_{i_1 \ldots i_m} \tag{7}$$

Since there are now n^m gain matrices, this generalization has the major inconvenience of vastly increasing the number of decision variables of the optimization problem. Moreover, a lot of the gains are actually superfluous, e.g., with $n = m = 2$, it is easily noticed that $K_{1,2}$ and $K_{2,1}$ play a similar role and could be replaced by a single gain of the form $L = (K_{1,2} + K_{2,1})/2$:

$$\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} h_{i_1} h_{i_2} K_{i_1,i_2} = h_1^2 K_{1,1} + 2h_1 h_2 L + h_2^2 K_{2,2} \tag{8}$$

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where h_1^2, $2h_1h_2$, and h_2^2 can be considered as new activation functions following the usual convex sum properties (2). The T-S literature has already addressed the redundancy of multi-sums in order to obtain relaxed LMI conditions of stability and stabilization by regrouping redundant terms [6], [8]. However, to the authors' surprise, no *explicit formula* was ever given in the T-S literature to rewrite the multi-sum (7) in a non-redundant manner, leaving this rewriting process to be handled either manually or algorithmically.

This paper is concerned with replacing the redundant multi-sum by its *explicit* non-redundant expression from the very beginning, skipping the usual regrouping and rewriting step. In doing so, the authors bring to light the geometric nature of the multi-sum relaxation, which turns out to be a simple Bézier interpolation scheme, a fact not discussed in the literature yet. The explicit Bézier writing reduces the number of gains from n^m to $(m+n-1)!/m!(n-1)!$ without hindering the capabilities of the control law or of the observer.

The geometric idea behind the writing introduced in this paper is the following: if the gain matrices $(K_i)_{1\leq i\leq n}$ are interpreted as the vertices of a "gain polytope", then, the approach of this paper consists in adding "control points" to the gain polytope which can deform its shape using a Bézier interpolation scheme. The shape of the gain polytope becomes "malleable" with a degree of flexibility given by the number of control points considered.

Note that other approaches exist to the stabilization and observation problem for T-S models, such as using more involved observers and PDC / non-PDC control laws [2], [3], [9], using dynamical controllers [10], [11], and modifying the Lyapunov function used [12], [13], [14]. Nevertheless, these approaches remain out of the scope of this paper.

The paper is organized as follows: in Section II, some definitions and notations are introduced. Section III provides two preliminary results on the Bernstein polynomials. Section IV contains the main results of this document, i.e. the LMI conditions allowing for the computation of the gain matrices of the Bézier controllers and observers. In this section, both Quadratic Lyapunov Functions (QLF) and NQLF are considered. Section V provides numerical examples of the previous LMI conditions, illustrating the reduced conservatism of the Bézier approach. Finally, some conclusions and perspectives are discussed in Section VI.

II. Definitions & Notations

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ stand resp. for the set of non-negative integers, real numbers and non-negative real numbers, and for $k,m\in\mathbb{N}$, $[\![k,m]\!]\triangleq\{r\in\mathbb{N}, k\leq r\leq m\}$. Let $i,j\in[\![1,n]\!]$

$$\mathbf{0}\triangleq(0,\dots,0)\in\mathbb{N}^n \tag{9}$$

$$\mathbf{1}_i\triangleq(0,\dots,0,\underbrace{1}_{i\text{-th coordinate}},0,\dots,0)\in\mathbb{N}^n \tag{10}$$

$$\delta_{i,j}\triangleq 1 \text{ if } i=j,\ 0 \text{ else} \tag{11}$$

$\mathbb{N}^n_m \triangleq \{\mathbf{k}\in\mathbb{N}^n : \sum_{i=1}^n k_i = m\}$, and $\Delta_{n-1}\triangleq\left\{\mathbf{X}\in\mathbb{R}^n_{\geq 0} : \sum_{i=1}^n X_i = 1\right\}$ stands for the $(n-1)$-simplex of $\mathbb{R}^n$. Let $\mathbf{k}\triangleq(k_1,\dots,k_n)\in\mathbb{N}^n_m$. $\binom{m}{\mathbf{k}}\triangleq\frac{m!}{k_1!\dots k_n!}$ stands for the multinomial coefficient. Given $\mathbf{X}\triangleq(X_1,\dots,X_n)\in\Delta_{n-1}$, $\mathbf{X}^{\mathbf{k}}$ denotes the monomial $X_1^{k_1}\dots X_n^{k_n}$, $\mathcal{B}^m_{\mathbf{k}}(\mathbf{X})\triangleq\binom{m}{\mathbf{k}}\mathbf{X}^{\mathbf{k}}$ stands for the $\mathbf{k}$-th multivariate Bernstein polynomial of degree m.

$\mathbb{R}^{n\times m}$ stands for the set of real matrices with n rows and m columns. Given $M\in\mathbb{R}^{n\times m}$, $M^\top$ stands for the transpose of M, and if $m=n$, then $\mathcal{H}(M)\triangleq M+M^\top$. Given a symmetric matrix $M=M^\top\in\mathbb{R}^{n\times n}$, M positive definite (resp. negative definite) is denoted $M\succ 0$ (resp. $M\prec 0$).

III. Preliminary Results

The Bézier interpolation scheme relies on weighting "control points" $(\Gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^n_m}$ by the multivariate Bernstein polynomials of degree m with parameters $\mathbf{X}\triangleq(X_1,\dots,X_n)$ belonging to the $(n-1)$-simplex. Formally, the interpolation of $(\Gamma_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^n_m}$ at $\mathbf{X}\in\Delta_{n-1}$ is given by $\Gamma(\mathbf{X}) = \sum_{\mathbf{i}\in\mathbb{N}^n_m}\mathcal{B}^m_{\mathbf{i}}(\mathbf{X})\Gamma_{\mathbf{i}}$. Note that they are as many multivariate Bernstein polynomial of degree m as they are elements in $\mathbb{N}^n_m$, a number given by counting the weak n-composition of m.

Lemma 1: $\mathbb{N}^n_m$ has $(m+n-1)!/m!(n-1)!$ elements.

Proof: See the "stars and bars" proof in [15] (p26). ■

The number of elements of $\mathbb{N}^n_m$ is compared below to the expression n^m for several values of (n,m), providing the amount of useless gains economized using a Bézier interpolation framework rather than a multi-sum.

(n,m)	$(m+n-1)!/m!(n-1)!$	n^m
$(3,3)$	10	27
$(3,4)$	15	81
$(3,5)$	21	243
$(6,3)$	56	216
$(6,4)$	126	1296
$(6,5)$	252	7776

Among the many properties of the multivariate Bernstein polynomials [16], two of them are crucial to this document, and are examined before proceeding to the main results.

Lemma 2: The multivariate Bernstein polynomials of degree $m\in\mathbb{N}$ satisfy the convex sum properties on the $(n-1)$-simplex

$$\forall\mathbf{X}\in\Delta_{n-1},\ \begin{cases}\forall\mathbf{k}\in\mathbb{N}^n_m : \mathcal{B}^m_{\mathbf{k}}(\mathbf{X})\geq 0\\ \sum_{\mathbf{i}\in\mathbb{N}^n_m}\mathcal{B}^m_{\mathbf{i}}(\mathbf{X})=1\end{cases} \tag{12}$$

Proof: The sign property follows from the definition of the $(n-1)$-simplex and of the multivariate Bernstein polynomials. The multinomial theorem provides the sum property $\sum_{\mathbf{i}\in\mathbb{N}^n_m}\binom{m}{\mathbf{i}}\mathbf{X}^{\mathbf{i}}=[X_1+\dots+X_n]^m=1$ ($\mathbf{X}\in\Delta_{n-1}$). ■

Lemma 3: Given the sets of control points $(\Gamma_{\mathbf{i},j})_{\mathbf{i}\in\mathbb{N}^n_m}$ with $j\in[\![1,n]\!]$, for all $\mathbf{X}\in\Delta_{n-1}$

$$\sum_{j=1}^n X_j\sum_{\mathbf{i}\in\mathbb{N}^n_m}\mathcal{B}^m_{\mathbf{i}}(\mathbf{X})\Gamma_{\mathbf{i},j}=\sum_{\mathbf{i}\in\mathbb{N}^n_{m+1}}\mathcal{B}^{m+1}_{\mathbf{i}}(\mathbf{X})\sum_{j=1}^n\frac{i_j}{m+1}\Gamma_{\mathbf{i}-\mathbf{1}_j,j} \tag{13}$$

Proof: Given $\mathbf{i}\in\mathbb{N}^n$, if there exists $k\in[\![1,n]\!]$ such that $i_k<0$, then it is considered that $\binom{m}{\mathbf{i}}=0$. The following equalities stand and prove (13)

$$\begin{aligned}
&\sum_{j=1}^{n} X_j \sum_{\mathbf{i}\in\mathbb{N}_m^n} \binom{m}{\mathbf{i}} \mathbf{X}^{\mathbf{i}} \Gamma_{\mathbf{i},j} = \sum_{\mathbf{i}\in\mathbb{N}_m^n} \sum_{j=1}^{n} \binom{m}{\mathbf{i}} \mathbf{X}^{\mathbf{i}+\mathbf{1}_j} \Gamma_{\mathbf{i},j} \\
&= \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \sum_{j=1}^{n} \binom{m}{\mathbf{i}-\mathbf{1}_j} \mathbf{X}^{\mathbf{i}} \Gamma_{\mathbf{i}-\mathbf{1}_j,j} \\
&= \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \binom{m+1}{\mathbf{i}} \mathbf{X}^{\mathbf{i}} \sum_{j=1}^{n} \binom{m}{\mathbf{i}-\mathbf{1}_j} \binom{m+1}{\mathbf{i}}^{-1} \Gamma_{\mathbf{i}-\mathbf{1}_j,j} \\
&= \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \binom{m+1}{\mathbf{i}} \mathbf{X}^{\mathbf{i}} \sum_{j=1}^{n} \frac{i_j}{m+1} \Gamma_{\mathbf{i}-\mathbf{1}_j,j}
\end{aligned}$$

■

Remark 1: This property can be viewed as a generalization of the "degree elevation property" of the multivariate Bernstein polynomials (see section 1.4 of [17]).

IV. The Bézier controllers and observers

This section deals with the LMI formulations of the optimization problems allowing for the computation of the set of gains $(K_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}_m^n}$ providing respectively:

- the exponential stability of (1) under the Bézier generalization of the PDC control law (3), as well as the exponential stability of the observation error between the state of (1) and the state of the Bézier generalization of the T-S observer (4);
- the exponential stability of (1) under the Bézier generalization of the non-PDC control law (5), as well as the exponential stability of the observation error between the state of (1) and the state of the Bézier generalization of the T-S observer (6).

These controllers and observers are based on a Bézier-simplex interpolation scheme, where the activation functions serve as the interpolating parameters belonging to the $(n-1)$-simplex, i.e. $\mathbf{h} \triangleq (h_1, \dots, h_n) \in \Delta_{n-1}$. The LMI formulations of the resulting optimization problems are given respectively by Theorem 1 (Bézier-PDC controller design) and Corollary 1 (Bézier-PDC observer design) for a QLF, and by Theorem 2 (Bézier-non-PDC controller design) and Corollary 2 (Bézier-non-PDC observer design) for NQLFs.

A. The Bézier-PDC approach

The following results hold for a Bézier-PDC control law

$$u(t) = \sum_{\mathbf{i}\in\mathbb{N}_m^n} \mathcal{B}_{\mathbf{i}}^m(\mathbf{h}) K_{\mathbf{i}} x(t) \quad (14)$$

and for its observer counterpart

$$\begin{cases} \dot{\hat{x}}(t) = & \sum_{j=1}^{n} h_j \left[A_j \hat{x}(t) + B_j u(t)\right] \\ & + \sum_{\mathbf{i}\in\mathbb{N}_m^n} \mathcal{B}_{\mathbf{i}}^m(\mathbf{h}) K_{\mathbf{i}} [\hat{y}(t) - y(t)] \\ \hat{y}(t) = & \sum_{j=1}^{n} h_j C_j \hat{x}(t) \end{cases} \quad (15)$$

considered with the QLF: $V(x) = x^\top P x$, with $P = P^\top \succ 0$.

Theorem 1: Given $m \in \mathbb{N}$, the system (1) is exponentially stable under the control law (14) if there exists $X = X^\top \succ 0$ and $M_{\mathbf{i}}$ such that the LMIs (16) are satisfied for all $\mathbf{i} \in \mathbb{N}_{m+1}^n$.

$$\sum_{j=1}^{n} \frac{i_j}{m+1} \mathcal{H}(A_j X + B_j M_{\mathbf{i}-\mathbf{1}_j}) \prec 0 \quad (16)$$

The controller gains are given by $K_{\mathbf{i}} = M_{\mathbf{i}} X^{-1}$.

Proof: The dynamic of the closed-loop system is given by

$$\dot{x}(t) = \sum_{j=1}^{n} h_j \left[A_j + \sum_{\mathbf{i}\in\mathbb{N}_m^n} \mathcal{B}_{\mathbf{i}}^m(\mathbf{h}) B_j K_{\mathbf{i}}\right] x(t) \quad (17)$$

which, thanks to Lemma 2 and Lemma 3, can be written as

$$\begin{aligned}
\dot{x}(t) &= \sum_{j=1}^{n} h_j \sum_{\mathbf{i}\in\mathbb{N}_m^n} \mathcal{B}_{\mathbf{i}}^m(\mathbf{h}) \left[A_j + B_j K_{\mathbf{i}}\right] x(t) \\
&= \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \mathcal{B}_{\mathbf{i}}^{m+1}(\mathbf{h}) \sum_{j=1}^{n} \frac{i_j}{m+1} [A_j + B_j K_{\mathbf{i}-\mathbf{1}_j}] x(t)
\end{aligned} \quad (18)$$

The usual results on quadratic stability for T-S models [4], [18] can finally be applied to the convex sum above. System (1) is exponentially stable under the control law (14) if the following conditions are satisfied

$$\forall \mathbf{i} \in \mathbb{N}_{m+1}^n : \sum_{j=1}^{n} \frac{i_j}{m+1} \mathcal{H}(P A_j + P B_j K_{\mathbf{i}-\mathbf{1}_j}) \prec 0 \quad (19)$$

the proof is concluded by a left and right multiplication of (19) by $X = P^{-1}$. ■

Corollary 1: Given $m \in \mathbb{N}$, the observation error between the state of (1) and the state of the observer (15) is exponentially stable if there exists $P = P^\top \succ 0$ and $M_{\mathbf{i}}$ such that the LMIs (20) are satisfied for all $\mathbf{i} \in \mathbb{N}_{m+1}^n$.

$$\sum_{j=1}^{n} \frac{i_j}{m+1} \mathcal{H}(P A_j + M_{\mathbf{i}-\mathbf{1}_j} C_j) \prec 0 \quad (20)$$

The observer gains are given by $K_{\mathbf{i}} = P^{-1} M_{\mathbf{i}}$.

Proof: The dynamic of the error $e(t) = \hat{x}(t) - x(t)$ is given by

$$\dot{e}(t) = \sum_{j=1}^{n} h_j \left[A_j + \sum_{\mathbf{i}\in\mathbb{N}_m^n} \mathcal{B}_{\mathbf{i}}^m(\mathbf{h}) K_{\mathbf{i}} C_j\right] e(t) \quad (21)$$

From here, the proof follows the same steps as for Theorem 1, without the final left and right multiplication by P^{-1}. ■

Remark 2: The separation principle stated in [19] and [20] still holds for this controller and observer design, hence it is possible to use the LMIs given above in order to compute an observer-based state feedback control law.

B. The Bézier-non-PDC approach

The previous results can be extended to a Bézier-non-PDC control law

$$u(t) = \left[\sum_{\mathbf{i}\in\mathbb{N}_m^n} \mathcal{B}_{\mathbf{i}}^m(\mathbf{h})K_{\mathbf{i}}\right] Q(\mathbf{h})x(t) \tag{22}$$

considered with the NQLF: $V(x) = x^\top Q(\mathbf{h})x$, as well as to its observer counterpart

$$\begin{cases} \dot{\hat{x}}(t) = & \sum_{j=1}^n h_j\left[A_j\hat{x}(t) + B_ju(t)\right] \\ & +Q(\mathbf{h})\sum_{\mathbf{i}\in\mathbb{N}_m^n} \mathcal{B}_{\mathbf{i}}^m(\mathbf{h})K_{\mathbf{i}}[\hat{y}(t) - y(t)] \\ \hat{y}(t) = & \sum_{j=1}^n h_jC_j\hat{x}(t) \end{cases} \tag{23}$$

considered with the NQLF: $V(x) = x^\top Q^{-1}(\mathbf{h})x$, where $Q^{-1}(\mathbf{h}) = \sum_{k=1}^n h_kP_k$ and $P_k = P_k^\top \succ 0$ for all $k \in [\![1,n]\!]$.

Theorem 2: Given $m \in \mathbb{N}$ and $\phi_k \in \mathbb{R}_{\geq 0}$ such that $|\dot{h}_k| \leq \phi_k$ for all $k \in [\![1,n]\!]$, the system (1) is exponentially stable under the control law (22) if there exists $P_k = P_k^\top \succ 0$ and $K_{\mathbf{i}}$ such that the LMIs (24) are satisfied for all $\mathbf{i} \in \mathbb{N}_{m+2}^n$

$$\sum_{j=1}^n \left[\phi_jP_j + \sum_{k=1}^n \frac{i_k(i_j - \delta_{j,k})}{(m+2)(m+1)}\mathcal{H}(T_{\mathbf{i}-\mathbf{1}_j-\mathbf{1}_k,j,k})\right] \prec 0 \tag{24}$$

where $T_{\mathbf{i},j,k} = [A_jP_k + B_jK_{\mathbf{i}}]$.

Proof: Thanks to Lemmas 2 and 3, the closed-loop dynamic of (1) with (22) is given by

$$\dot{x}(t) = \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \mathcal{B}_{\mathbf{i}}^{m+1}(\mathbf{h})\tilde{A}_{\mathbf{i}}x(t) \tag{25}$$

where $\tilde{A}_{\mathbf{i}} = \sum_{j=1}^n \frac{i_j}{m+1}[A_j + B_jK_{\mathbf{i}-\mathbf{1}_j}Q(\mathbf{h})]$. The usual results on nonquadratic stability for T-S models [5], [18] provide the exponential stability of (1) under the control law (22) if $R \prec 0$, where

$$R = \dot{Q}(\mathbf{h}) + \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \mathcal{B}_{\mathbf{i}}^{m+1}(\mathbf{h})\mathcal{H}(Q(\mathbf{h})\tilde{A}_{\mathbf{i}}) \tag{26}$$

By congruence, $R \prec 0$ holds if and only if $S(\mathbf{h}) = Q^{-1}(\mathbf{h})RQ^{-1}(\mathbf{h}) \prec 0$, with

$$S(\mathbf{h}) = -[\dot{Q^{-1}}](\mathbf{h}) + \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \mathcal{B}_{\mathbf{i}}^{m+1}(\mathbf{h})\mathcal{H}(\tilde{A}_{\mathbf{i}}Q^{-1}(\mathbf{h})) \tag{27}$$

where $[\dot{Q^{-1}}](\mathbf{h}) = -Q^{-1}(\mathbf{h})\dot{Q}(\mathbf{h})Q^{-1}(\mathbf{h}) = \sum_{k=1}^n \dot{h}_kP_k$. Moreover, the following equalities hold:

$$\begin{aligned} &\sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \mathcal{B}_{\mathbf{i}}^{m+1}(\mathbf{h})\tilde{A}_{\mathbf{i}}Q^{-1}(\mathbf{h}) \\ &= \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \mathcal{B}_{\mathbf{i}}^{m+1}(\mathbf{h})\sum_{j=1}^n \frac{i_j}{m+1}[A_jQ^{-1}(\mathbf{h}) + B_jK_{\mathbf{i}-\mathbf{1}_j}] \\ &= \sum_{k=1}^n h_k \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \mathcal{B}_{\mathbf{i}}^{m+1}(\mathbf{h})\Gamma_{\mathbf{i},k} \end{aligned} \tag{28}$$

with $\Gamma_{\mathbf{i},k} = \sum_{j=1}^n \frac{i_j}{m+1}[A_jP_k + B_jK_{\mathbf{i}-\mathbf{1}_j}]$. Thanks to Lemma 3, this provides

$$\begin{aligned} (28) &= \sum_{\mathbf{i}\in\mathbb{N}_{m+2}^n} \mathcal{B}_{\mathbf{i}}^{m+2}(\mathbf{h})\sum_{k=1}^n \frac{i_k}{m+2}\Gamma_{\mathbf{i}-\mathbf{1}_k,k} \\ &= \sum_{\mathbf{i}\in\mathbb{N}_{m+2}^n} \mathcal{B}_{\mathbf{i}}^{m+2}(\mathbf{h})\sum_{j=1}^n\sum_{k=1}^n \frac{i_k(i_j-\delta_{j,k})}{(m+2)(m+1)}T_{\mathbf{i}-\mathbf{1}_j-\mathbf{1}_k,j,k} \end{aligned} \tag{29}$$

with $T_{\mathbf{i},j,k} = [A_jP_k + B_jK_{\mathbf{i}}]$. Finally, $|\dot{h}_k| \leq \phi_k$ provides the LMIs of the theorem and concludes its proof. ■

Corollary 2: Given $m \in \mathbb{N}$ and $\phi_k \in \mathbb{R}_{\geq 0}$ such that $|\dot{h}_k| \leq \phi_k$ for all $k \in [\![1,n]\!]$, the observation error between the state of (1) and the state of the observer (23) is exponentially stable if there exists $P_k = P_k^\top \succ 0$ and $K_{\mathbf{i}}$ such that the LMIs (30) are satisfied for all $\mathbf{i} \in \mathbb{N}_{m+2}^n$

$$\sum_{j=1}^n \left[\phi_jP_j + \sum_{k=1}^n \frac{i_k(i_j - \delta_{j,k})}{(m+2)(m+1)}\mathcal{H}(T_{\mathbf{i}-\mathbf{1}_j-\mathbf{1}_k,j,k})\right] \prec 0 \tag{30}$$

where $T_{\mathbf{i},j,k} = [P_kA_j + K_{\mathbf{i}}C_j]$.

Proof: The error dynamic $e(t) = \hat{x}(t) - x(t)$ is given by

$$\dot{e}(t) = \sum_{\mathbf{i}\in\mathbb{N}_{m+1}^n} \mathcal{B}_{\mathbf{i}}^{m+1}(\mathbf{h})\tilde{A}_{\mathbf{i}}e(t) \tag{31}$$

where $\tilde{A}_{\mathbf{i}} = \sum_{j=1}^n \frac{i_j}{m+1}\left[A_j + Q(\mathbf{h})K_{\mathbf{i}-\mathbf{1}_j}C_j\right]$. From here, the proof follows the same steps as for Theorem 2, without the left and right multiplication by $Q^{-1}(\mathbf{h})$. ■

V. Illustrative Examples

A. The Bézier-PDC approach

To illustrate the conservatism reduction brought by the Bézier-PDC controller design, the following T-S model (taken from [21], [8]) is considered

$$\mathcal{S}_{(a,b)}: \quad \dot{x}(t) = \sum_{i=1}^3 h_i[A_i(a)x(t) + B_i(b)u(t)] \tag{32}$$

with $A_1 = \begin{bmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{bmatrix}$, $A_3 = \begin{bmatrix} -a & -4.33 \\ 0 & 0.05 \end{bmatrix}$, $B_1 = [1 \ \ 0]^\top$, $B_2 = [8 \ \ 0]^\top$, $B_3 = [6-b \ \ -1]^\top$, together with the QLF: $V(x) = x^\top X^{-1}x$, with $X = X^\top \succ 0$.

In the following, the stabilization problem of $\mathcal{S}_{(a,b)}$ is considered at several values of $(a,b) \in \mathbb{R}^2$ for the control laws $u(t) = K_m(\mathbf{h})x(t)$, with $m \in [\![0,2]\!]$ and

$$K_0(\mathbf{h}) = K_{000} \tag{33}$$

$$K_1(\mathbf{h}) = h_1K_{100} + h_2K_{010} + h_3K_{001} \tag{34}$$

$$\begin{aligned} K_2(\mathbf{h}) = &h_1^2K_{200} + h_2^2K_{020} + h_3^2K_{002} + 2h_1h_2K_{110} \\ &+ 2h_1h_3K_{101} + 2h_2h_3K_{011} \end{aligned} \tag{35}$$

For $m = 2$, $n^m - (m+n-1)!/m!(n-1)! = 3$ redundant gain matrices have been economized compared to the usual multi-sum approach, i.e. 6 useless decision variables. It is recalled that:

$$\mathbb{N}_0^3 = \{(0,0,0)\}, \quad \mathbb{N}_1^3 = \{(1,0,0);(0,1,0);(0,0,1)\}, \tag{36}$$

$$\mathbb{N}_2^3 = \{(2,0,0);(0,2,0);(0,0,2);(1,1,0);(1,0,1);\\ (0,1,1)\}, \tag{37}$$

$$\mathbb{N}_3^3 = \{(3,0,0);(0,3,0);(0,0,3);(2,1,0);(2,0,1);\\ (1,2,0);(0,2,1);(1,0,2);(0,1,2);(1,1,1)\} \tag{38}$$

Following from the results of Theorem 1, the LMI conditions to compute the Bézier-PDC feedback $K_2(\mathbf{h})$ are given by

$$\forall i \in [\![1,3]\!] : \ \mathcal{H}(A_i X + B_i M_{000+2\cdot\mathbf{1}_i}) \prec 0 \tag{39}$$

$$\frac{1}{3}\mathcal{H}(2[A_1X + B_1M_{110}] + [A_2X + B_2M_{200}]) \prec 0 \tag{40}$$

$$\frac{1}{3}\mathcal{H}(2[A_1X + B_1M_{101}] + [A_3X + B_3M_{200}]) \prec 0 \tag{41}$$

$$\frac{1}{3}\mathcal{H}([A_1X + B_1M_{020}] + 2[A_2X + B_2M_{110}]) \prec 0 \tag{42}$$

$$\frac{1}{3}\mathcal{H}(2[A_2X + B_2M_{011}] + [A_3X + B_3M_{020}]) \prec 0 \tag{43}$$

$$\frac{1}{3}\mathcal{H}([A_1X + B_1M_{002}] + 2[A_3X + B_3M_{101}]) \prec 0 \tag{44}$$

$$\frac{1}{3}\mathcal{H}([A_2X + B_2M_{002}] + 2[A_3X + B_3M_{011}]) \prec 0 \tag{45}$$

$$\frac{1}{3}\mathcal{H}\left(\sum_{i=1}^{3}[A_iX + B_iM_{111-\mathbf{1}_i}]\right) \prec 0 \tag{46}$$

The feedback $K_0(\mathbf{h})$ and the PDC feedback $K_1(\mathbf{h})$ are computed with the LMI conditions given by Theorem 1 as well.

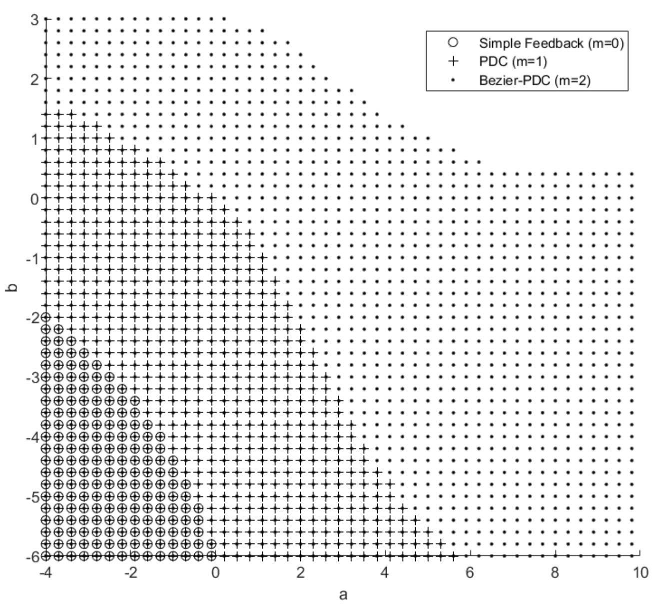

Fig. 1. Stabilizability (a,b)-regions of $\mathcal{S}_{(a,b)}$ with a PDC control law computed with the LMI conditions of Theorem 1

Figure 1 illustrates the (a,b)-regions for which a solution is found to the LMIs given above. The (a,b)-region gets larger as m increases. This demonstrates the increased capabilities of the Bézier-PDC control law compared to the usual PDC approach.

B. The Bézier-non-PDC approach

To illustrate the conservatism reduction brought by the Bézier-non-PDC controller design, the following T-S model is considered

$$\mathcal{T}_{(\phi_1,\phi_2)}: \quad \dot{x}(t) = \sum_{i=1}^{2} h_i[A_ix(t) + B_iu(t)] \tag{47}$$

with

$$A_1 = \begin{bmatrix} 2 & -10 & 3 & 1 & 5 \\ 2 & 0 & 1 & 2 & 4 \\ -1 & 0 & -5 & 0 & -2 \\ 1 & 0 & 5 & 0 & -1 \\ -1 & 5 & 4 & 3 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ -2 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 5 & 2 & -1 & 1 \\ 1 & 2 & 1 & -2 & -1 \\ -1 & 0 & -10 & -1 & -1 \\ 1 & 0 & -10 & 1 & -1 \\ 4 & 5 & -1 & -2 & 5 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 1 & -1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$$

under the assumptions $|\dot{h}_1| \leq \phi_1$ and $|\dot{h}_2| \leq \phi_2$, together with the NQLF: $V(x) = x^\top[h_1P_1 + h_2P_2]^{-1}x$, with symmetric $P_1, P_2 \succ 0$.

Now, the stabilization problem of $\mathcal{T}_{(\phi_1,\phi_2)}$ is considered at several values of $(\phi_1,\phi_2) \in \mathbb{R}^2_{\geq 0}$ for the control laws $u(t) = K_m(\mathbf{h})[h_1P_1 + h_2P_2]^{-1}x(t)$, with $m \in [\![0,3]\!]$ and

$$K_0(\mathbf{h}) = K_{00} \tag{48}$$

$$K_1(\mathbf{h}) = h_1K_{10} + h_2K_{01} \tag{49}$$

$$K_2(\mathbf{h}) = h_1^2K_{20} + 2h_1h_2K_{11} + h_2^2K_{02} \tag{50}$$

$$K_3(\mathbf{h}) = h_1^3K_{30} + 3h_1^2h_2K_{21} + 3h_1h_2^2K_{12} + h_2^3K_{03} \tag{51}$$

For $m = 3$, $n^m - (m+n-1)!/m!(n-1)! = 4$ redundant gain matrices have been economized compared to the usual multi-sum approach, i.e. 40 useless decision variables. It is recalled that:

$$\mathbb{N}_0^2 = \{(0,0)\}, \quad \mathbb{N}_1^2 = \{(1,0);(0,1)\}, \tag{52}$$

$$\mathbb{N}_2^2 = \{(2,0);(1,1);(0,2)\}, \tag{53}$$

$$\mathbb{N}_3^2 = \{(3,0);(2,1);(1,2);(0,3)\}, \tag{54}$$

$$\mathbb{N}_4^2 = \{(4,0);(3,1);(2,2);(1,3);(0,4)\} \tag{55}$$

$$\mathbb{N}_5^2 = \{(5,0);(4,1);(3,2);(2,3);(1,4);(0,5)\} \tag{56}$$

Let $R(\phi) = -\phi_1P_1 - \phi_2P_2$. Following from the results of Theorem 2, the LMI conditions to compute the Bézier-non-PDC feedback $K_3(\mathbf{h})[h_1P_1 + h_2P_2]^{-1}$ are given by

$$\forall i \in [\![1,2]\!] : \ \mathcal{H}(A_iP_i + B_iK_{00+3\cdot\mathbf{1}_i}) \prec R(\phi) \tag{57}$$

$$\frac{1}{5}\mathcal{H}(3[A_1P_1 + B_1K_{21}] + [A_1P_2 + B_1K_{30}] \\ + [A_2P_1 + B_2K_{30}]) \prec R(\phi) \tag{58}$$

$$\frac{1}{5}\mathcal{H}([A_1P_2+B_1K_{03}]+[A_2P_1+B_2K_{03}] \\ +3[A_2P_2+B_2K_{12}]) \prec R(\phi) \quad (59)$$

$$\frac{1}{10}\mathcal{H}(3[A_1P_1+B_1K_{12}]+3[A_1P_2+B_1K_{21}] \\ +3[A_2P_1+B_2K_{21}]+[A_2P_2+B_2K_{30}]) \prec R(\phi) \quad (60)$$

$$\frac{1}{10}\mathcal{H}([A_1P_1+B_1K_{03}]+3[A_1P_2+B_1K_{12}] \\ +3[A_2P_1+B_2K_{12}]+3[A_2P_2+B_2K_{21}]) \prec R(\phi) \quad (61)$$

The feedback $K_0(\mathbf{h})[h_1P_1+h_2P_2]^{-1}$, the non-PDC feedback $K_1(\mathbf{h})[h_1P_1+h_2P_2]^{-1}$ and the Bézier-non-PDC feedback $K_2(\mathbf{h})[h_1P_1+h_2P_2]^{-1}$ are computed with the LMI conditions given by Theorem 2 as well.

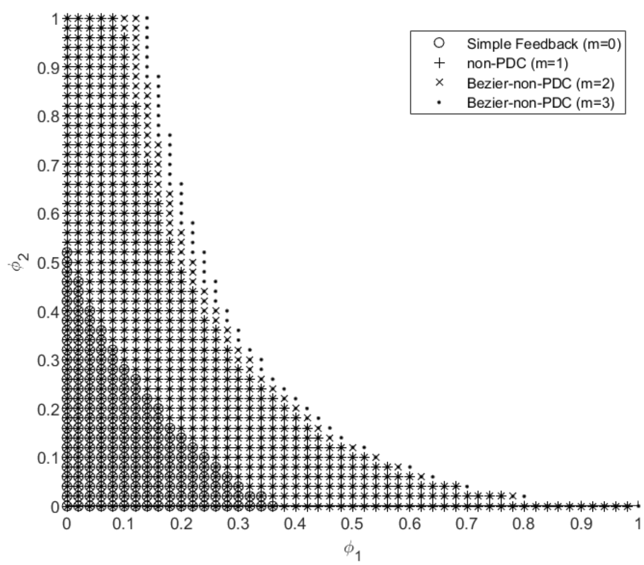

Fig. 2. Stabilizability (ϕ_1,ϕ_2)-regions of $\mathcal{T}_{(\phi_1,\phi_2)}$ with a non-PDC control law computed with the LMI conditions of Theorem 2

Figure 2 illustrates the (ϕ_1,ϕ_2)-regions for which a solution is found to the LMIs given above. The (ϕ_1,ϕ_2)-region gets larger as m increases. This demonstrates the increased capabilities of the Bézier-non-PDC control law compared to the usual non-PDC approach. The region difference is however less pronounced than in the previous example. The authors explain this variation by the T-S model only having two local models, which makes the stabilization problem easily solvable. Given a NQLF, the Bézier-non-PDC approach solves some of the stabilization problems with solutions that were not already solved by the usual non-PDC approach, and there are not many of them.

VI. Conclusions and Perspectives

In this paper, new less conservative controllers and observers for T-S models have been provided. Their designs are based on a Bézier interpolation scheme of gain matrices. Simple LMI formulations of the resulting stabilization problems have been provided for QLFs and NQLFs. It should be noted that the LMI conditions given in this paper could be relaxed further, e.g. by using the results from [8]. The extension of this Bézier approach remains to be investigated for discrete-time T-S models, as well as for T-S models with an unmeasurable scheduling vector.

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