Bounding the Trajectories of Continuous-Time LPV Systems with Parameters known in Real Time

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Abstract: Predicting the exact future of a Linear Parameter-Varying (LPV) system with its parameter exclusively known in real time is by definition an impossible task. In particular, it is difficult to quantify the error induced by a Zero-Order Hold (ZOH) discretization of the parameter which is not verified in practice (e.g. when the parameter depends on the states, inputs or outputs of the continuous-time system). Under a Lipschitz assumption, this paper upper bounds — in terms of uncertain matrices — the greatest possible discrepancy between the real future of the system and its estimate based on the last known values of the input and parameter. This not only upper bounds the error due to the ZOH discretization, but also provides sufficient conditions for controllability and observability of the system in the near future by bounding its Gramians.

Keywords: LPV Systems, Zero-Order Hold Discretization, Sampled-Data Systems, Uncertain Systems, Observability Gramian, Controllability Gramian, State-transition Matrix, Product Integration.

1. INTRODUCTION

The study of nonlinear systems under a Linear Parameter-Varying (LPV) representation is motivated by the similarity between the LPV framework and the linear one, which facilitates controller synthesis for these nonlinear systems. The LPV representation was indeed introduced in a gain-scheduling control context [Shamma and Athans (1990, 1991); Rugh and Shamma (2000)]. Under the LPV representation, the nonlinearities are generally written in terms of a scheduling parameter θ which is assumed to be known or estimated in real time.

However, such models are subject to some issues regarding their discretization [Tóth et al. (2008, 2010)]. If a controller is synthetized for a continuous-time LPV system [Packard and Becker (1992); Scherer (1996)], its implementation is generally sampled, and the dynamic of θ is neglected during the sampling period (Figure 1): this sampled-data issue has already been discussed for example in [Tan et al. (2002); Ramezanifar et al. (2012)]. Moreover, if the controller is synthetized for a discrete-time LPV system directly [Packard (1994); Apkarian and Gahinet (1995); Jungers et al. (2011)], in practice, the discrete-time representation is often derived from a continuous-time LPV model using a Zero-Order Hold (ZOH) assumption on θ. Yet, this continuous-time LPV model is itself a representation of a continuous-time non-linear system. In this context, the dynamic of θ is once again neglected during the sampling period.

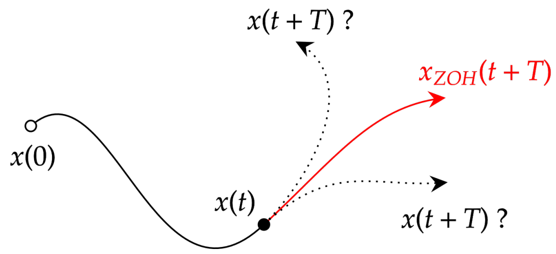

Fig. 1. Potential discrepancy between the future trajectory of the system and the trajectory under the ZOH assumption on θ

To overcome this issue, the present paper proposes to bound the future state trajectory of a continuous-time LPV system in terms of norm-bounded uncertain matrices (Figure 2). This result derives from a Lipschitz assumption made on the time variation of the parameter θ. The bounded variation rate assumption on θ is not new in the LPV literature. For example, it is often used in the stability analysis of such models [Hoffmann (2016)] or in the model predictive control synthesis setting [Suzukia and Sugie (2006); Besselmann et al. (2009); Li and Xi (2010); Jungers et al. (2011)]. However, to the authors' knowledge, it has not been discussed in order to bound the future state trajectory in a continuous-time setting before.

The paper is organised as follows: first, in Section 2, the main inequalities on the state and input matrices are given, allowing the encapsulation of the system future trajectory. Then, in Section 3, these inequalities are applied to obtain an estimation of the Gramians, providing Gramian-based sufficient controllability and observability conditions in the near future for LPV systems. Section 2 and 3 end with illustrative examples. Finally, some perspectives are discussed in Section 4.

2. BOUNDING THE FUTURE STATE TRAJECTORY

This section quantifies the greatest possible discrepancy between the real future of an LPV system with bounded parameter variations (Lipschitz assumption), and an artificially constructed prediction of the future for which the real-time parameter and the input are being held constant.

To obtain these results, a discussion on the state transition matrix of continuous-time Linear Time-Varying (LTV) must first be conducted, which leads to the introduction of the product integral tool. Uncertain matrices are then introduced in the system to represent its divergence compared to the estimated future, and upper bounds are given on their norm using properties of the product integral. These upper bounds correspond to an evaluation of the error due to the ZOH discretization of a continuous-time LPV system for which the ZOH assumption on the parameter cannot be verified in pratice, e.g. when the parameter depends on the states, inputs or outputs of the continuous-time system. Sharpness of the bounds on the state matrix for a near future is finally illustrated by a simple example.

2.1 The state transition matrix as a product integral

Given a continuous-time LTV system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{1}$$

where $A : \mathbb{R} \to \mathbb{K}^{n\times n}$, $B : \mathbb{R} \to \mathbb{K}^{n\times m}$ and $C : \mathbb{R} \to \mathbb{K}^{l\times n}$ are continuous functions whose values are known in advance at all time $t \in \mathbb{R}$, and where $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. The trajectories of the system are given, using the state transition matrix $\Phi(t, t_0)$, by

$$x(t) = \Phi(t, t_0)\, x(t_0) + \int_{t_0}^{t} \Phi(t, s)\, B(s)u(s)ds \tag{2}$$

where for all $t_1, t_2 \in \mathbb{R}$ such that $t_1 \le t_2$, the value $\Phi(t_2, t_1)$ is expressed in terms of the Peano–Baker series (which is a Picard iteration, sometimes called the Volterra series)

$$\Phi(t_2, t_1) = \sum_{k=0}^{+\infty} J_k(t_2, t_1) \tag{3}$$

where the J_k are recursively defined by

$$\begin{cases} J_0(t_2, t_1) = I \\ \forall k \in \mathbb{N},\ J_{k+1}(t_2, t_1) = \displaystyle\int_{t_1}^{t_2} A(s)J_k(s, t_1)ds \end{cases} \tag{4}$$

with I the identity matrix. Instead of using this series, this paper reuses Volterra's idea to consider this state-transition matrix as an infinite composition of infinitesimal linear transformations [Volterra and Hostinský (1938)]. The following is indeed intuitively true

$$\Phi(t_2, t_1) = \lim_{\delta t \to 0^+} e^{A(t_2)\delta t} e^{A(t_2-\delta t)\delta t} \dots e^{A(t_1)\delta t} \tag{5}$$

and corresponds to a multiplicative version of the Riemann integral which does not commute. This construction has already been defined rigorously and is commonly referred to as the product integral [Dollard and Friedman (1984); Slavík (2007)]. It is denoted in this article in the following way:

$$\Phi(t_2, t_1) = \prod_{t_1}^{t_2} e^{A(s)ds} \tag{6}$$

Among the many properties of the product integral, two of them are being crucial to this document.

Lemma 1. (Duhamel's Formula). Let $E, F : \mathbb{R} \to \mathbb{K}^{p\times p}$ be two continuous functions. For all $t_1, t_2 \in \mathbb{R}$

$$\begin{aligned} &\prod_{t_1}^{t_2} e^{E(s)ds} - \prod_{t_1}^{t_2} e^{F(s)ds} = \\ &\int_{t_1}^{t_2} \prod_{\tau}^{t_2} e^{F(s)ds} \left(E(\tau) - F(\tau)\right) \prod_{t_1}^{\tau} e^{E(s)ds} d\tau \end{aligned} \tag{7}$$

See Theorem 5.1 of [Dollard and Friedman (1984)].

Lemma 2. Let $E : \mathbb{R} \to \mathbb{K}^{p\times p}$ be a continuous function. For all $t_1, t_2 \in \mathbb{R}$ such that $t_1 \le t_2$

$$\left\| \prod_{t_1}^{t_2} e^{E(s)ds} \right\| \le e^{\int_{t_1}^{t_2} \mu(E(s))ds} \tag{8}$$

where $\mu(\cdot)$ denotes the logarithmic norm induced by the operator norm on the left part of the inequality

$$\mu(E) = \lim_{h \to 0^+} \frac{\|I + hE\| - 1}{h} \tag{9}$$

This second result is a consequence of Grönwall's lemma [Desoer and Haneda (1972)].

2.2 Main Results

The expression of a continuous-time LPV system is generally given by

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))u(t) \\ y(t) &= C(\theta(t))x(t) \end{aligned} \tag{10}$$

where A, B and C affinely depend on the parameter θ, which is exclusively known in real time. This expression has, up to the real time assumption on θ, the same nature as its LTV counterpart (1):

$$A(t) \equiv A(\theta(t)),\ B(t) \equiv B(\theta(t)),\ C(t) \equiv C(\theta(t)) \tag{11}$$

Hence, in the following, the LTV notations are kept in the LPV context, due to their clarity and concision.

Knowing the values of A and B at time t and assuming the control u is held constant between t and $t + T$, an estimation of the state of the system (10) at a time $t + T$ can generally be carried out using the ZOH assumption on θ (for all $\tau \in [t, t + T]$, $A(\tau) = A(t)$, $B(\tau) = B(t)$):

$$x_{ZOH}(t + T) = e^{TA(t)}x(t) + \left(\int_0^T e^{sA(t)}ds\right) B(t)u(t) \tag{12}$$

The idea of this section consists in estimating the error made with this ZOH assumption, thus upper bounding the norm of Δ_A and Δ_B, two matrices modelling the error and taken such that the following equality is satisfied:

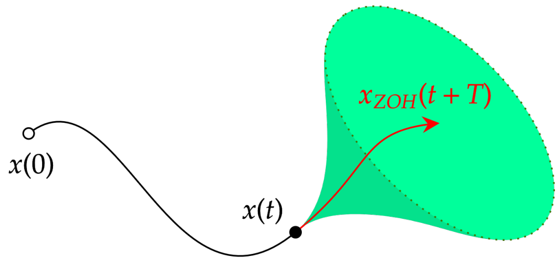

Fig. 2. Bounds on the future system trajectory: $x(t+T)$ belongs to the green set with a dotted boundary line

$$x(t+T) = \left(e^{TA(t)} + \Delta_A\right) x(t) + \left(\left(\int_0^T e^{sA(t)}ds\right) B(t) + \Delta_B\right) u(t) \tag{13}$$

A schematic illustration of the bounds is given in Figure 2.

Assumption 1. A and B are respectively L_A-Lipschitz and L_B-Lipschitz on the interval $[t, t+T]$ [1] and $\sigma \in \mathbb{R}$ is an upper bound to the logarithmic norm of the state matrix:

$$\sigma \geq \sup_{\tau \in [t,t+T]} \mu(A(\tau)) \tag{14}$$

with L_A, L_B and $\mu(\cdot)$ induced by the norm used on Δ_A, Δ_B and $B(t)$.

The Lipschitz assumption on A and B is similar to the usual assumption of θ having a bounded rate of variations, and the values for L_A and L_B can easily be derived from (11) by the bounds on the first derivative of θ. Moreover, in the case of the spectral norm, the value of σ can easily be deduced from the bounds on θ thanks to the Lemma 1 of (Jungers et al., 2017).

Theorem 1. (Main Inequalities). Under Assumption 1, the two following inequalities hold:

$$\begin{aligned} \|\Delta_A\| &\leq \frac{1}{2} L_A T^2 e^{\sigma T} \\ \|\Delta_B\| &\leq \frac{1}{\sigma^2} L_B \left(e^{\sigma T} - \sigma T - 1\right) \\ &+ \frac{1}{2\sigma^3} L_A \left(\left(\sigma^2 T^2 - 2\right) e^{\sigma T} + \sigma T + 2\right) \|B(t)\| \end{aligned} \tag{15}$$

Proof. Equations (2) and (13) provide

$$\begin{aligned} \Delta_A &= \prod_t^{t+T} e^{A(s)ds} - e^{TA(t)} \\ \Delta_B &= \int_t^{t+T} \left(\prod_\tau^{t+T} e^{A(s)ds} B(\tau) - e^{(\tau-t)A(t)} B(t)\right) d\tau \end{aligned} \tag{16}$$

which can be written as

$$\begin{aligned} \Delta_A &= \prod_t^{t+T} e^{A(s)ds} - \prod_t^{t+T} e^{A(t)ds} \\ \Delta_B &= \Delta_{B,1} + \Delta_{B,2} B(t) \\ &= \int_t^{t+T} \prod_\tau^{t+T} e^{A(s)ds} \left(B(\tau) - B(t)\right) d\tau \\ &+ \int_t^{t+T} \left(\prod_\tau^{t+T} e^{A(s)ds} - \prod_\tau^{t+T} e^{A(t)ds}\right) d\tau \, B(t) \end{aligned} \tag{17}$$

Lemma 1 yields

$$\begin{aligned} \Delta_A &= \int_t^{t+T} e^{(t+T-\tau)A(t)} \left(A(\tau) - A(t)\right) \prod_t^\tau e^{A(s)ds} d\tau \\ \Delta_{B,2} &= \\ &\int_t^{t+T} \int_\tau^{t+T} e^{(t+T-\delta)A(t)} (A(\delta) - A(t)) \prod_\tau^\delta e^{A(s)ds} d\delta d\tau \end{aligned} \tag{18}$$

Under the Lipschitz assumptions, thanks to Lemma 2 and the upper bound on the logarithmic norm, the following inequalities hold

$$\begin{aligned} \|\Delta_A\| &\leq L_A \int_t^{t+T} (\tau - t) e^{(t+T-\tau)\mu(A(t)) + \int_t^\tau \mu(A(s))ds} d\tau \\ &\leq L_A e^{\sigma T} \int_0^T \tau d\tau \leq \frac{1}{2} L_A T^2 e^{\sigma T} \end{aligned} \tag{19}$$

similarly

$$\begin{aligned} \|\Delta_{B,2}\| &\leq L_A \int_t^{t+T} e^{\sigma(t+T-\tau)} \int_\tau^{t+T} (\delta - t) d\delta d\tau \\ &\leq \frac{1}{2} L_A \int_t^{t+T} \left(T^2 - (\tau - t)^2\right) e^{\sigma(t+T-\tau)} d\tau \\ &\leq \frac{1}{2} L_A \int_0^T \left(T^2 - \tau^2\right) e^{\sigma(T-\tau)} d\tau \\ &\leq \frac{1}{2\sigma^3} L_A \left(\left(\sigma^2 T^2 - 2\right) e^{\sigma T} + \sigma T + 2\right) \end{aligned} \tag{20}$$

and finally

$$\begin{aligned} \|\Delta_{B,1}\| &\leq L_B \int_t^{t+T} (\tau - t) e^{\sigma(t+T-\tau)} d\tau \\ &\leq L_B \int_0^T (T - \tau) e^{\sigma\tau} d\tau \\ &\leq \frac{1}{\sigma^2} L_B \left(e^{\sigma T} - \sigma T - 1\right) \end{aligned} \tag{21}$$

which concludes the proof. ■

Corollary 1. (Exact discretization). If the Lipschitz hypotheses hold for A and B in general, and if the logarithmic norm of A is upper bounded in general as well, then the system (13) can be viewed as the exact discretization of the LPV system (10) at a sampling period T, where Δ_A and Δ_B represent bounded uncertainties.

This discretization is exact in the sense that it takes into account the uncertainties introduced by the ZOH discretization of the parameter θ, which — to the authors' knowledge — were previously not taken into account in the literature [Tóth et al. (2008, 2010)]. This discretization can therefore be used in order to synthesize a sampled controller or observer for the continuous-time LPV system.

[1] Very similar results are obtainable if the Lipschitz hypotheses are weakened to α-Hölder conditions, or if A and B are piecewise continuous and bounded.

2.3 Illustrative example of the state matrix inequality and of its sharpness

The sharpness of the previous inequality on Δ_A is illustrated for small T through the study of an example. Consider the LPV system

$$\dot{x}(t) = \theta(t)x(t) \tag{22}$$

where $\theta : \mathbb{R} \to \mathbb{C}$ is assumed to be a L_A-Lipschitz continuous parameter only known in real time, with initial value $\theta(0) = \sigma \in \mathbb{R}$ and such that for all $t \in \mathbb{R}$, $\mu(\theta(t)) \leq \sigma$. Let $T \geq 0$. From the initial value of θ, Theorem 1 yields

$$\begin{aligned} x(T) &= \left(e^{\sigma T} + \Delta_A\right) x(0) \\ \text{with } |\Delta_A| &\leq \frac{1}{2} L_A T^2 e^{\sigma T} \end{aligned} \tag{23}$$

After observing θ during the time span $[0, T]$, suppose its evolution was given by:

$$\forall t \in [0, T],\, \theta(t) = \sigma + \mathbf{i}tL_A \tag{24}$$

where $\mathbf{i}$ denotes the imaginary unit. This provides:

$$\begin{aligned} x(T) &= e^{\int_0^T (\sigma + \mathbf{i}sL_A)ds} x(0) \\ &= e^{\sigma T + \frac{1}{2}\mathbf{i}L_A T^2} x(0) \end{aligned} \tag{25}$$

the real value taken by the uncertainty Δ_A can then be retrieved

$$\begin{aligned} |\Delta_A| &= \left| e^{\sigma T + \frac{1}{2}\mathbf{i}L_A T^2} - e^{\sigma T} \right| \\ &= \sqrt{2 - 2\cos\left(\frac{1}{2}L_A T^2\right)} e^{\sigma T} \\ &= 2\left|\sin\left(\frac{1}{4}L_A T^2\right)\right| e^{\sigma T} \leq \frac{1}{2}L_A T^2 e^{\sigma T} \end{aligned} \tag{26}$$

and it almost coincides with the previous upper bound for small values of T, demonstrating the sharpness of the inequality on Δ_A for a near future. Precisely:

$$|\Delta_A| \underset{T\to 0^+}{=} \frac{1}{2}L_A T^2 e^{\sigma T} + O(T^6) \tag{27}$$

3. CONTROLLABILITY AND OBSERVABILITY IN THE NEAR FUTURE

In this section, the previous results are applied to find sufficient conditions of controllability and observability in the near future of an LPV system, which could be useful to obtain guarantees on the convergence of finite-time controllers and observers. Such conditions are based on an estimation of the controllability and observability Gramian where the estimation error is upper bounded with the help of Theorem 1.

As an introduction to this section, the classical results on controllability and observability of an LTV system (1) given by their Gramian are recalled:

Lemma 3. The pair $(A(t), B(t))$ (resp. $(A(t), C(t))$) of the system (1) is controllable (resp. observable) at time t_1 if and only if there exists a finite time $t_2 > t_1$ such that the controllability Gramian (28) (resp. the observability Gramian (29)) is positive definite, which is denoted $W_c(t_2, t_1) \succ 0$ (resp. $W_o(t_2, t_1) \succ 0$).

$$W_c(t_2, t_1) = \int_{t_1}^{t_2} \Phi(t_2, \tau)B(\tau)B^*(\tau)\Phi^*(t_2, \tau)d\tau \tag{28}$$

$$W_o(t_2, t_1) = \int_{t_1}^{t_2} \Phi^*(t_2, \tau)C^*(\tau)C(\tau)\Phi(t_2, \tau)d\tau \tag{29}$$

This result can be found for example in [Chen et al. (2004)]. Given a matrix $M \in \mathbb{K}^{p\times q}$, M^* denotes its conjugate transpose.

3.1 Inequalities for the LPV Gramian estimation

This section considers the LPV system (10), where B and C are assumed to be continuous functions whose values are known in advance at all time $t \in \mathbb{R}$. The parameter θ providing the values of A in real time is still denoted implicitly. An estimation of the controllability Gramian of the system between t_1 and t_2, with $t \leq t_1 < t_2$ can be carried out using:

$$\hat{W}_c(t_2, t_1) = \int_{t_1}^{t_2} e^{(t_2-\tau)A(t)}B(\tau)B^*(\tau)e^{(t_2-\tau)A^*(t)}d\tau \tag{30}$$

resp. for the observability Gramian:

$$\hat{W}_o(t_2, t_1) = \int_{t_1}^{t_2} e^{(t_2-\tau)A^*(t)}C^*(\tau)C(\tau)e^{(t_2-\tau)A(t)}d\tau \tag{31}$$

The idea of this section consists in estimating the upper bounds of $\Delta_{W_c}(t_2, t_1)$ and $\Delta_{W_o}(t_2, t_1)$ in (32):

$$\begin{aligned} W_c(t_2, t_1) &= \hat{W}_c(t_2, t_1) + \Delta_{W_c}(t_2, t_1) \\ W_o(t_2, t_1) &= \hat{W}_o(t_2, t_1) + \Delta_{W_o}(t_2, t_1) \end{aligned} \tag{32}$$

These two matrices define the error made in the Gramian calculations when A cannot be known in advance and thus $A(\tau) = A(t)$ is assumed for $\tau \in [t_1, t_2]$.

Theorem 2. (Gramian Inequalities). If A is L_A-Lipschitz on the interval $[t, t_2]$ [2], then, for all $t_1 \in [t, t_2)$ the two following inequalities hold:

$$\begin{aligned} &\|\Delta_{W_c}(t_2, t_1)\|_2 \leq \\ &\quad L_A \int_{t_1}^{t_2} (t_2 - \tau)(t_2 + \tau - 2t)e^{2\sigma(t_2-\tau)}\|B(\tau)\|_2^2 d\tau \end{aligned} \tag{33}$$

$$\begin{aligned} &\|\Delta_{W_o}(t_2, t_1)\|_2 \leq \\ &\quad L_A \int_{t_1}^{t_2} (t_2 - \tau)(t_2 + \tau - 2t)e^{2\sigma(t_2-\tau)}\|C(\tau)\|_2^2 d\tau \end{aligned} \tag{34}$$

where σ denotes an upper bound to the logarithmic norm of the state matrix:

$$\sigma \geq \sup_{\tau\in[t,t_2]} \mu_2(A(\tau)) \tag{35}$$

with $\|\cdot\|_2$ the spectral norm and $\mu_2(\cdot)$ the logarithmic norm induced by the spectral norm.

Proof. The proof only focuses on the controllability Gramian, the reasoning being similar for the observability Gramian. First, for any matrices $E, F \in \mathbb{K}^{p\times q}$, the following identity holds:

$$2\left(EE^* - FF^*\right) = (E+F)(E-F)^* + (E-F)(E+F)^* \tag{36}$$

hence

$$\|EE^* - FF^*\|_2 \leq \|E - F\|_2\|E + F\|_2 \tag{37}$$

Now, applying this inequality to (32) using (28) and (30) provides

$$\|\Delta_{W_c}(t_2, t_1)\|_2 \leq \int_{t_1}^{t_2} \|\Delta_1(\tau)\|_2\|\Delta_2(\tau)\|_2\|B(\tau)\|_2^2 d\tau \tag{38}$$

[2] Again, similar results are obtainable if the Lipschitz hypothesis is weakened to an α-Hölder condition, or if A is piecewise continuous and bounded.

with

$$\begin{aligned}\Delta_1(\tau) &= \prod_{\tau}^{t_2} e^{A(s)ds} - e^{(t_2-\tau)A(t)} \\ \Delta_2(\tau) &= \prod_{\tau}^{t_2} e^{A(s)ds} + e^{(t_2-\tau)A(t)}\end{aligned} \tag{39}$$

which can be written as

$$\begin{aligned}\Delta_1(\tau) &= \prod_{\tau}^{t_2} e^{A(s)ds} - \prod_{\tau}^{t_2} e^{A(t)ds} \\ \Delta_2(\tau) &= \prod_{\tau}^{t_2} e^{A(s)ds} + \prod_{\tau}^{t_2} e^{A(t)ds}\end{aligned} \tag{40}$$

Lemma 1 yields

$$\Delta_1(\tau) = \int_{\tau}^{t_2} e^{(t_2-\delta)A(t)} \left(A(\delta) - A(t)\right) \prod_{\tau}^{\delta} e^{A(s)ds} d\delta \tag{41}$$

Under the Lipschitz assumptions, thanks to Lemma 2 and the upper bound on the logarithmic norm, the following inequalities hold:

$$\begin{aligned}\|\Delta_1(\tau)\|_2 &\leq L_A e^{\sigma(t_2-\tau)} \int_{\tau}^{t_2} (\delta - t) d\delta \\ &\leq \frac{1}{2} L_A (t_2-\tau)(t_2+\tau-2t) e^{\sigma(t_2-\tau)}\end{aligned} \tag{42}$$

and

$$\|\Delta_2(\tau)\|_2 \leq 2e^{\sigma(t_2-\tau)} \tag{43}$$

which concludes the proof. ■

3.2 Gramian-based sufficient controllability and observability conditions in the near future for LPV systems

Thanks to the previous upper bound on the Gramian estimation error, sufficient conditions on the controllability and observability of the LPV system (10) in the future can be exhibited after the proof of the following lemma.

Lemma 4. Given $P, P_1, P_2, \Delta \in \mathbb{K}^{p\times p}$ four Hermitian matrices, if

$$P_1 \preceq P \preceq P_2 \tag{44}$$

then

$$P_1 - \|\Delta\|_2 I_n \preceq P + \Delta \preceq P_2 + \|\Delta\|_2 I_n \tag{45}$$

The implication is also true for the strict partial order $\prec$.

Proof. The proof only focuses on the left part of (45), the reasoning being similar up to a sign change for its right part. The proof is also similar for the strict partial order $\prec$. First notice that by definition:

$$\begin{aligned}\|\Delta\|_2 =& \sup_{x\neq 0} \frac{\|\Delta x\|_2}{\|x\|_2} \\ =& \sup_{x,y\neq 0} \frac{|y^*\Delta x|}{\|x\|_2\|y\|_2} \geq \sup_{x\neq 0} \frac{|x^*\Delta x|}{\|x\|_2^2}\end{aligned} \tag{46}$$

Moreover

$$P_1 \preceq P \tag{47}$$

can be re-written as

$$\forall x \neq 0,\ x^* P_1 x \leq x^* P x \tag{48}$$

that is to say

$$\forall x \neq 0,\ \frac{1}{\|x\|_2^2} x^* P_1 x \leq \frac{1}{\|x\|_2^2} x^* P x \tag{49}$$

hence

$$\forall x \neq 0,\ \frac{1}{\|x\|_2^2} x^* P_1 x - \sup_{y\neq 0} \frac{|y^*\Delta y|}{\|y\|_2^2} \leq \frac{1}{\|x\|_2^2} x^* (P+\Delta) x \tag{50}$$

therefore

$$\forall x \neq 0,\ x^* P_1 x - \|\Delta\|_2 \|x\|_2^2 \leq x^*(P+\Delta)x \tag{51}$$

which is equivalent to

$$\forall x \neq 0,\ x^* \left(P_1 - \|\Delta\|_2 I_n\right) x \leq x^* \left(P + \Delta\right) x \tag{52}$$

i.e.

$$P_1 - \|\Delta\|_2 I_n \preceq P + \Delta \tag{53}$$

■

Theorem 3. Despite not having access to the value of $(A(t), B(t))$ (resp. $(A(t), C(t))$) in the system (10) at a future time t_1, the pair $(A(t), B(t))$ (resp. $(A(t), C(t))$) is controllable (resp. observable) at time t_1 if there exists a finite time $t_2 > t_1$ such that the estimate of the controllability Gramian defined by (30) (resp. the estimate of the observability Gramian defined by (31)) satisfies (54) (resp. (55)).

$$\hat{W}_c(t_2, t_1) - m_c I_n \succ 0 \tag{54}$$

$$\hat{W}_o(t_2, t_1) - m_o I_n \succ 0 \tag{55}$$

where m_c and m_o verify respectively:

$$m_c \geq \|\Delta_{W_c}(t_2, t_1)\|_2 \tag{56}$$

$$m_o \geq \|\Delta_{W_o}(t_2, t_1)\|_2 \tag{57}$$

Proof. The proof only focuses on the controllability Gramian, the reasoning being similar for the observability Gramian. From Lemma 3, we know that $(A(t), B(t))$ is controllable at time t_1 if and only if there exists a finite time $t_2 > t_1$ such that

$$W_c(t_2, t_1) \succ 0 \tag{58}$$

that is to say, thanks to (32), such that

$$\hat{W}_c(t_2, t_1) + \Delta_{W_c}(t_2, t_1) \succ 0 \tag{59}$$

Lemma 4 then provides:

$$\begin{aligned}\hat{W}_c(t_2, t_1) + \Delta_{W_c}(t_2, t_1) \succ \\ \hat{W}_c(t_2, t_1) - \|\Delta_{W_c}(t_2, t_1)\|_2 I_n\end{aligned} \tag{60}$$

hence

$$W_c(t_2, t_1) \succ \hat{W}_c(t_2, t_1) - m_c I_n \tag{61}$$

meaning $(A(t), B(t))$ is controllable at time t_1 if there exists a finite time $t_2 > t_1$ such that (54) holds, which achieves the proof for the controllability part. ■

3.3 Illustrative example

Consider the LPV system

$$\dot{x}(t) = \theta(t)x(t) + u(t) \tag{62}$$

where $\theta : \mathbb{R} \to \mathbb{C}$ is assumed to be a L_A-Lipschitz continuous parameter only known in real time, such that for all $t \in \mathbb{R}$, $\mu(\theta(t)) \leq \sigma$, with $L_A = 1$, $\sigma = 1/2$ and $\theta(0) = 0$. Of course, such a system is controllable on $[0, +\infty)$. The point of this example is to show that the condition of Theorem 3, despite being conservative, is sufficient to arrive at this conclusion.

Using (30) with $t = 0$ and $\theta(t) = 0$ to estimate the controllability Gramian

$$\hat{W}_c(t_2, t_1) = \int_{t_1}^{t_2} d\tau = t_2 - t_1 \tag{63}$$

and using the results of Theorem 2 with $t = 0$ to upper bound the estimation error

$$\|\Delta_{W_c}(t_2, t_1)\|_2 \leq \int_{t_1}^{t_2} (t_2 - \tau)(t_2 + \tau)e^{t_2 - \tau} d\tau$$
$$= 2(t_2 + 1) + \left(t_2^2 - t_1^2 - 2(t_1 + 1)\right) e^{t_2 - t_1} \quad (64)$$

Theorem 3 states that the system is controllable on $[0, +\infty)$ if for all $t_1 \in [0, +\infty)$ there exists $t_2 > t_1$ such that $\alpha(t_1, t_2) > 0$ where $\alpha(t_1, t_2)$ is defined by:

$$\alpha(t_1, t_2) = \left(2(t_1 + 1) + t_1^2 - t_2^2\right) e^{t_2 - t_1} - t_2 - t_1 - 2 \quad (65)$$

It is easy to check that $\alpha(t_1, t_1) = 0$. Moreover

$$\frac{\partial \alpha}{\partial t_2}(t_1, t_1) = 1 \quad (66)$$

hence, for all $t_1 \in [0, +\infty)$, there exists an $\epsilon > 0$ such that $\alpha(t_1, t_1 + \epsilon) > 0$. This demonstrates that the system (62) is indeed always controllable on $[0, +\infty)$.

4. CONCLUSION AND PERSPECTIVES

In this paper, bounds are given on uncertain matrices in order to bound for the future trajectory of a continuous-time LPV system with its parameter exclusively known in real time. These bounds are then used to demonstrate controllability and observability of such a system in the future, but other applications of these bounds remain to be investigated. For example, a fault detection scheme can be imagined to check if the observed future respects the previously constructed bounds. The results of Section 3 could also be useful to obtain guarantees on the convergence of finite-time controllers and observers. The extension to a continuous-time setting of the discrete-time MPC techniques for LPV systems with a bounded rate of variation [Suzukia and Sugie (2006); Besselmann et al. (2009); Li and Xi (2010); Jungers et al. (2011)] may also rely on these results.

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