Interval state estimation for uncertain polytopic systems.

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ABSTRACT
The aim of this article is the state estimation of uncertain polytopic dynamic systems. The parametric uncertainties affecting the system are time varying, unknown and bounded with known bounds. The objective is to determine the state estimates consisting in the smallest interval containing the real state value caused by the parametric uncertainties. This set will be characterized by the lower and upper bounds of the state trajectory. Given the uncertainty bounds, the set can be computed by a direct simulation of the system but a more accurate estimation is obtained with a Luenberger-type observer, fed with the system measurements. The proposed observer is designed to minimize the interval width of the estimates. The observer gains are obtained by solving an optimization problem under LMI constraints. The efficiency of the proposed approach is illustrated by numerical examples.

1. Introduction

Observer design is a key issue in engineering since the system state variables are needed for controller design or system supervision but all of these variables may not be directly measured for economical or physical reasons. In this case, based on the system model and some measurements, an observer can be designed to reconstruct the state values. Since (Kalman, 1960; Luenberger, 1971), an abundant literature is devoted to observer design for different system classes, e.g. (Besançon, 2007; Boubaker et al., 2019; Lendek, Guerra, Babuska, & De Schutter, 2010). Since physical systems are always affected by uncertainties, such as modeling errors, parameters uncertainties, measurement noises, they may influence the performance analysis and the control design. Therefore, the way to deal with uncertainties is a key issue. In a disturbed context, it is well known that the so-called Kalman filter may provide a state estimation, if some statistical properties of the noises affecting the system are known (Kalman, 1960). When no statistical properties of the disturbances is available, the interval observer design is an efficient alternative to the robust estimation approach aiming at minimizing the influence of the uncertainties on the estimated state trajectory. Interval observers are dynamical systems designed to provide, at each time instant, estimations of the upper and lower bounds of the state of a system affected by bounded uncertainties e.g. (Efimov & Raïssi, 2016; Moisan, Bernard, & Gouzé, 2009; Raïssi, Efimov, & Zolghadri, 2012). Their main advantage is that, provided that the uncertainty bounds are known, the

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interval estimation results in a guaranteed estimate in the sense that, for all possible values of the uncertainties, the real state trajectory belongs to the estimated envelope defined by the upper and lower bounds. Many interval analysis methods are proposed to evaluate the bounds of the state of a system described by interval models. One may refer to the pioneer work of (Moore, 1963) which was then followed by some works on structural properties (Faydasicoka & Arikb, 2013; Haoa, Lianga, & Roy, 2015), stability analysis (Pastravanu & Matcovschi, 2010; Udom, 2012), parameter estimation (Ramdani, Raïssi, Candau, & Ibos, 2005; Shao & Su, 2010; C. Wang, Li, & Guo, 2015), state estimation (Ben Chabane, Stoica Maniu, Alamo, Camacho, & Dumur, 2014; Chisci, Garulli, & Zappa, 1996), diagnosis (Martinez-Sibaja et al., 2011) and control (Park, Kwon, & Park, 2015; Polyakov, Efimov, Perruquetti, & Richard, 2013). Some papers addressed the design of interval observer with structural constraints, as in (C. Wang & Sheng, 2012) where positive observer for positive linear continuous system is proposed. With the same constraints, observer design for discrete-time switched systems has been presented in (Guo & Zhu, 2017b; He & Xie, 2016).

As pointed in the survey (Efimov & Raïssi, 2016), interval observer design has now reached a certain maturity for linear time-invariant (LTI) systems as well as time-delay systems or linear parameter varying (LPV) systems. As an illustration of this maturity, one may consider the works on unknown input (UI) observer design and the observer-based fault detection and isolation (FDI). The interval observer design for invariant and uncertain continuous-time systems with UI is discussed in (Ellero, Gucik-Derigny, & Henry, 2016), while the discrete-time case is addressed in (Meyer, Ichalal, & Vigneron, 2018; Robinson, Marzat, & Raïssi, 2017). In (Rotondo, Cristofaro, Johansen, Nejjari, & Puig, 2018), the case of continuous systems with variable parameters is tackled. In the field of FDI, the detection and estimation of actuator failure (considered an UI) is proposed using an interval observer of the state and the UI in (C. M. Garcia, Puig, Astorga-Zaragoza, & Osorio-Gordillo, 2018; Guo, Zhu, Zhu, Hu, & Yang, 2017). The approach proposed in (Guo & Zhu, 2017a) is slightly different: two cascading interval observers are used to estimate the system state and then, with a step delay, the UI. In (Zarch, Puig, Poshtan, & Shoorehdeli, 2018), is proposed a combined use of interval observers and viability theory for fault detection. The joint diagnosis/control problem is addressed in (Lamouchi, Raïssi, Amairi, & Aoun, 2018) where a linear state feedback is designed to compensate the impact of actuator faults on system performance by stabilizing the closed-loop system using interval observers. In (Zhang & Yang, 2017), the situation is a little different since FDI is performed on interconnected systems with unknown but bounded interconnection functions.

The main remaining problems in interval estimation are the extension of the existing results to systems with parametric uncertainties (in opposition to additive uncertain inputs) and to nonlinear systems. For instance, in (Cacace, Germani, & Manes, 2015; Mazenc & Bernard, 2010) only linear systems are considered and in (Z. Wang, Lim, & Shen, 2018) LTI systems with additive disturbances are under study. In (Ichalal, Marx, Maquin, & Ragot, 2018), polytopic uncertainties are dealt with but the considered models are only linear whereas in the present paper we consider nonlinear models affected by polytopic uncertainties. More precisely, in (Ichalal et al., 2018), when the uncertainties are null the considered systems become LTI, whereas in the present paper, even for null uncertainties, the considered systems are nonlinear (see Section 6 for an illustration of the brought improvement). The nonlinear case is tackled in (Efimov, Perruquetti, Raïssi, & Zolghadri, 2013; Efimov & Raïssi, 2016; C. Garcia, Puig, & Zaragoza, 2017; Menasria, Bouras, & Debbache, 2017; Raïssi et al., 2012; Thabet, Raïssi, Combastel, Efimov, & Zolghadri, 2014) since interval observers are designed for

LPV, Takagi-Sugeno (T-S) or time-delay systems, but no parametric uncertainties are encompassed in the model: only additive noises are considered. Moreover, in (Efimov, Perruquetti, et al., 2013; Efimov, Raïssi, Chebotarev, & Zolghadri, 2013; Raïssi et al., 2012; Thabet et al., 2014) a time-invariant or time-varying state transformation is needed in order to make the state estimation error Metzler or cooperative. In (Langowski & Brdys, 2017) the system is time varying and affected by uncertainties, but some restrictive assumptions are made: it is supposed to be cooperative, with a special structure of measured outputs and the state bounds need to be a priori known. Interval observer for uncertain nonlinear systems are proposed in (Efimov, Raïssi, et al., 2013), but once again a prior state transformation is needed and the observer gains are not optimized. Recently, interval estimation of uncertain LPV continuous-time systems has been proposed in (Y. Wang, Bevly, & Rajamani, 2015).

The aim of the present paper is to propose an interval observer for discrete-time nonlinear systems with bounded parametric uncertainties, without any prior state transformation and with an optimization of the observer gains in order to improve the estimation accuracy. The nonlinearity of the system is modeled by a polytopic structure. A polytopic linear model (PLM) is an efficient approach of nonlinear modeling, defined by a time varying convex interpolation between a set of submodels (Takagi & Sugeno, 1985). The motivation for using PLM mainly relies on their structural similarity with linear models allowing to use classical tools of the linear theory and also on their property of universal approximator (Lendek et al., 2010; Tanaka & Wang, 2001). Furthermore, the PLM formalism covers a large variety of models such as LPV systems, T-S systems, switched systems, etc. In the present paper, the considered uncertainties are bounded parametric uncertainties affecting the state equation. The first contribution of the present paper is to derive the state envelope of an uncertain nonlinear system. The second contribution is the design of an interval observer providing lower and upper bounds of the system state trajectory. The observer gains are optimized in order to obtain an accurate state estimation by solving an optimization problem under LMI constraints. One should note that, contrarily to most existing works on interval observer, no positivity or cooperativity property is imposed, neither any state transformation is needed to recover such properties.

The paper is organized as follows. The problem statement, the useful assumptions and the objective of the paper are developed in the second section. The computation of a state envelope for scalar or vector uncertain PLM is detailed in the third section. The obtained envelopes are PLM systems without uncertain terms but encompassing an absolute value term. In the fourth section, the observer structure is given and its gains are obtained by solving an LMI optimization problem. The fifth section is devoted to the extension of the previous results to PLM with unmeasurable premise variables. The sixth section presents a numerical example that clearly illustrates the contribution of the proposed observer compared to the state envelope of the third and fourth sections. Finally, some concluding remarks are given in the last section.

2. Problem statement

In order to design an observer for an uncertain nonlinear system, a model of this system is needed. As widely known, the PLM formalism (also known as Takagi-Sugeno or T-S formalism (Takagi & Sugeno, 1985)), consisting in a time-varying combination of linear sub-models, is an efficient modeling of any nonlinear system with bounded nonlinearities. An analytical and systematic procedure to transform a nonlinear system by rewriting it into a T-S form, without any loss of information is known as the sector nonlinearity transformation (Nagy, Mourot, Marx, Ragot, & Schutz, 2010; Tanaka & Wang, 2001). In the present paper, we therefore consider that the uncertain nonlinear system to observe is modeled by a PLM system with bounded parametric uncertainties. The following notations will be used hereafter.

Notation 1. *In the remaining of the paper, indexes $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, p\}$ are respectively used to indicate the discrete time and a submodel number. The strict or non strict inequalities denoted by $<$, $>$, $\leq$ and $\geq$ are componentwise when applied to vectors or matrices. The symbols $P \succ 0$ (resp. $P \prec 0$) is a linear matrix inequality and means that the matrix P is symmetric positive (resp. negative) definite. The absolute value of $\bullet$ is denoted $|\bullet|$ and is componentwise when applied to a vector or a matrix. The sign function is denoted $sign(\bullet)$. The transpose of a matrix P is denoted P^T. The matrices I_n and 0_n respectively denote the identity and null matrices of $\mathbb{R}^{n\times n}$. A block diagonal matrix with the blocks P_1, P_2, ..., P_n on its diagonal is denoted $diag(P_1, P_2, \ldots, P_n)$. For any square matrix M, $\mathcal{S}(M)$ is defined by $\mathcal{S}(M) = M + M^T$. The terms induced by symmetry are denoted by $*$, for instance $I_{2n} = \begin{pmatrix} I_n & * \\ 0_n & I_n \end{pmatrix}$.*

2.1. System structure

The discrete time representation (1) is considered, where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{n_u}$ the input, $y \in \mathbb{R}^{n_y}$ the measured output, A_k and B are matrices of appropriate dimensions. The matrices $\bar{A}_\ell$ are the certain part of the state matrices, whereas the matrices $\Delta_{\ell,k}$ stand for the uncertain part.

$$\left\{ \begin{array}{lcl} x_{k+1} & = & A_k\, x_k + B\, u_k \\ A_k & = & \sum\limits_{\ell=1}^{p} \mu_{\ell,k}(\rho_k) \left(\bar{A}_\ell + \Delta_{\ell,k}\right) \\ y_k & = & C\, x_k \end{array} \right. \tag{1}$$

The weighting functions $\mu_{\ell,k}(\rho_k)$ depend on measured parameter vector ρ_k and satisfy the so-called convex sum property (Tanaka & Wang, 2001)

$$\left\{ \begin{array}{l} 0 \leq \mu_{\ell,k} \leq 1 \quad \forall k \in \mathbb{N}, \text{ and } \ell = 1, ..., p \\ \sum_{\ell=1}^{p} \mu_{\ell,k} = 1, \quad \forall k \in \mathbb{N} \end{array} \right. \tag{2}$$

For the sake of clarity, in (2) and in the remaining of the paper, the weighting functions will be expressed without (ρ_k). Also for the sake of clarity, parametric uncertainties $\Delta_{\ell,k}$ are assumed to only affect the state matrix, but similar uncertainties affecting the B and C matrices are straightforward extensions of the proposed approach.

Hypothesis 1. *For $\ell = 1, \ldots, p$, the parametric uncertainties $\Delta_{\ell,k}$ are assumed to be bounded with known bounds, according to (3), where the inequalities are to be understood component-wise.*

$$-\Delta_\ell \leq \Delta_{\ell,k} \leq \Delta_\ell, \qquad \Delta_\ell > 0, \quad \forall k \in \mathbb{N} \tag{3}$$

Hypothesis 2. *The null matrix of the same dimensions as those of A_k does not belong to the interval matrix $[\mathcal{A}_k - \bar{\Delta}_k \quad \mathcal{A}_k + \bar{\Delta}_k]$ where $\mathcal{A}_k$ and $\bar{\Delta}_k$ are respectively defined* *by* $\mathcal{A}_k = \sum_{\ell=1}^{p} \mu_{\ell,k} \bar{A}_\ell$ *and* $\bar{\Delta}_k = \sum_{\ell=1}^{p} \mu_{\ell,k} \Delta_{\ell,k}$*. Put in other words, the upper and lower bounds of each entry of* A_k *have the same sign at every time instant* k*.*

Remark 1. *The structure* (1) *can be obtained by polytopic transformation of a nonlinear system by building the weighting functions with all the nonlinearities depending on measured variables, and confining the parametric uncertainties and the nonlinearities depending on unmeasured state variables in the uncertain terms. It allows to avoid unmeasured parameters in the weighting functions and thus eases the observer design.*

2.1.1. Illustrative example

Let us consider the nonlinear uncertain system described by

$$\begin{aligned} x_{1,k+1} &= a_{1,k} cos(x_{1,k} u_k) x_{2,k} \\ x_{2,k+1} &= (2 + a_{2,k} sin(x_{2,k})) x_{1,k} - x_{2,k} + u_k \\ y_k &= x_{1,k} \end{aligned} \tag{4}$$

where $a_{1,k}$ and $a_{2,k}$ are uncertain parameters of the form $a_{i,k} = \bar{a}_i + \delta_{i,k}$ $(i = 1, 2)$, where $\bar{a}_i$ is known and $\delta_{i,k}$ represent the uncertainties. The system (4) can then be expressed as follows

$$x_{k+1} = \left(\begin{bmatrix} 0 & \bar{a}_1 \cos(x_{1,k} u_k) \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & \delta_{1,k} \cos(x_{1,k} u_k) \\ a_{2,k} \sin(x_{2,k}) & 0 \end{bmatrix} \right) x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \tag{5}$$

Considering the bounded parameter $\rho_k = \cos(x_{1,k} u_k) = \cos(y_k u_k) \in [-1 \ 1]$, the following polytopic form of the system (4) is obtained

$$x_{k+1} = \sum_{\ell=1}^{2} \mu_{\ell,k} \left(A_\ell + \Delta_{\ell,k}\right) x_k + Bu \tag{6}$$

where $\mu_{1,k} = \frac{\rho_k + 1}{2}$, $\mu_{2,k} = \frac{1 - \rho_k}{2}$ and

$$\begin{aligned} &A_1 = \begin{bmatrix} 0 & \bar{a}_1 \\ 2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -\bar{a}_1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ &\Delta_{1,k} = \begin{bmatrix} 0 & \delta_{1,k} \\ a_{2,k} \sin(x_{2,k}) & 0 \end{bmatrix}, \quad \Delta_{2,k} = \begin{bmatrix} 0 & -\delta_{1,k} \\ a_{2,k} \sin(x_{2,k}) & 0 \end{bmatrix} \end{aligned}$$

Notice that this system can also be expressed in a polytopic form with 4 submodels by considering a second parameter defined by $\sin(x_{2,k})$. Unfortunately, since it depends on an unmeasured state, the observer design becomes more difficult to tackle and it is more efficient to consider this term as an uncertainty included in the matrix Δ_k.

2.2. Objective

For a given PLM (1) and the knowledge of the uncertain parameter bounds Δ_k, the objective is to build the upper and lower bounds of the state trajectory $\underline{x}_k$ and $\overline{x}_k$ containing all the possible values of the state variables corresponding to the values of the bounded uncertain parameters. These bounds may be determined by calculating the different possible state trajectories but it leads to a heavy calculus burden and the obtained interval width may be large.

The first goal of the present paper is to avoid the evaluation of all the possible values of the parametric uncertainties and the corresponding state trajectories. To do so, an analytical expression of the state bounds is defined and only the upper and lower extremal trajectories of the state envelope are recursively calculated, as done in the linear case in (Ichalal et al., 2018). It will be proved that all the possible state trajectories are included in this envelope. The drawback is that the width of this envelope may be large. To overcome this drawback, the second goal is to design an interval observer providing the lower and upper bounds of the system state trajectory. Based on a polytopic form of the uncertain nonlinear system, the observer gains are determined to minimize the interval width using the $\mathcal{L}_2$ approach and an LMI optimization.

3. Computing the state envelope of an uncertain PLM

3.1. The scalar case

To begin with, the simple case of a first order PLM (i.e. with $n = 1$) is studied in order to be then generalized to the PLM of any orders.

The interval form of an autonomous first order PLM with p submodels is:

$$\left\{ \begin{aligned} x_{k+1} &= a_k\, x_k \\ a_k &= \sum_{\ell=1}^{p} \mu_{\ell,k} \left(\bar{a}_\ell + \delta_{\ell,k}\right) \end{aligned} \right. \tag{7}$$

where the functions $\mu_{\ell,k}$ satisfy (2) and where the uncertainties are bounded as follows:

$$-\delta_\ell \leq \delta_{\ell,k} \leq \delta_\ell, \qquad \delta_\ell > 0 \tag{8}$$

According to the Hypothesis 2, zero does not belong to the interval $[a_\ell - \delta_\ell \quad a_\ell + \delta_\ell]$, thus the parameter lower and upper bounds $a_\ell - \delta_{\ell,k}$ and $a_\ell + \delta_{\ell,k}$ are of the same sign.

Based on the upper bound of the parametric uncertainty δ_ℓ, the aim is to determine the lower and upper bounds of the state $\underline{x}_k$ and $\overline{x}_k$, such that at each time instant, the following inequalities hold:

$$\underline{x}_k \quad \leq x_k \quad \leq \overline{x}_k \tag{9}$$

In order to formulate the state trajectory of the system (7) as an interval, the sign of the uncertain term a_k should be taken into account. Two situations must be considered. If $\bar{a}_\ell + \delta_{\ell,k} > 0$, according to Hypothesis 2, it holds $a_\ell > 0$ and it follows:

$$(\bar{a}_\ell + \delta_{\ell,k})\, \underline{x}_k \quad \leq (\bar{a}_\ell + \delta_{\ell,k})\, x_k \quad \leq (\bar{a}_\ell + \delta_{\ell,k})\, \overline{x}_k \tag{10}$$

From

$$\left\{ \begin{array}{llll} -\delta_\ell \mid \underline{x}_k \mid & \leq \delta_{\ell,k}\, \underline{x}_k & \leq \delta_\ell \mid \underline{x}_k \mid \\ -\delta_\ell \mid \overline{x}_k \mid & \leq \delta_{\ell,k}\, \overline{x}_k & \leq \delta_\ell \mid \overline{x}_k \mid \end{array} \right. \tag{11}$$

the inequalities (10) hold, for all possible values of the current uncertainties $\delta_{\ell,k}$, if :

$$\bar{a}_\ell\, \underline{x}_k - \delta_\ell \mid \underline{x}_k \mid \quad \leq (\bar{a}_\ell + \delta_{\ell,k})\, x_k \quad \leq \bar{a}_\ell\, \overline{x}_k + \delta_\ell \mid \overline{x}_k \mid \tag{12}$$

If $\bar{a}_\ell + \delta_{\ell,k} < 0$ (or equivalently, if $\bar{a}_\ell < 0$), it follows :

$$(\bar{a}_\ell + \delta_{\ell,k})\, \overline{x}_k \quad \leq (\bar{a}_\ell + \delta_{\ell,k})\, x_k \quad \leq (\bar{a}_\ell + \delta_{\ell,k})\, \underline{x}_k \tag{13}$$

From (11), (13) holds, for all possible values of the current uncertainties $\delta_{\ell,k}$, if :

$$\bar{a}_\ell\, \overline{x}_k - \delta_\ell \mid \overline{x}_k \mid \quad \leq (\bar{a}_\ell + \delta_{\ell,k})\, x_k \quad \leq \bar{a}_\ell\, \underline{x}_k + \delta_\ell \mid \underline{x}_k \mid \tag{14}$$

Gathering the two situations described in (12) and (14), the bounds of $(\bar{a}_\ell + \delta_{\ell,k})\, x_k$ at the time instant $k+1$ are obtained as:

$$\begin{aligned}
&\frac{1}{2}\left[\bar{a}_\ell\, (\underline{x}_k + \overline{x}_k) - \delta_\ell(\mid \underline{x}_k \mid + \mid \overline{x}_k \mid) + sign(\bar{a}_\ell)\, (\bar{a}_\ell\, (\underline{x}_k - \overline{x}_k) - \delta_\ell(\mid \underline{x}_k \mid - \mid \overline{x}_k \mid))\right] \\
&\qquad\qquad\qquad\qquad \leq (\bar{a}_\ell + \delta_{\ell,k})\, x_k \leq \\
&\frac{1}{2}\left[\bar{a}_\ell\, (\underline{x}_k + \overline{x}_k) + \delta_\ell(\mid \underline{x}_k \mid + \mid \overline{x}_k \mid) + sign(\bar{a}_\ell)\, (\bar{a}_\ell\, (\overline{x}_k - \underline{x}_k) + \delta_\ell(\mid \overline{x}_k \mid - \mid \underline{x}_k \mid))\right]
\end{aligned} \tag{15}$$

One should note that from Hypothesis 2, $\bar{a}_\ell + \delta_{\ell,k}$ cannot be null, and thus (12) and (14) cover all the possible cases. Finally, from (15) and taking into account all the parameters $\bar{a}_\ell$ involved in the state dynamic (7), the state envelope is defined by:

$$\left\{\begin{aligned}
\underline{x}_{k+1} &= \frac{1}{2}\sum_{\ell=1}^{p}\mu_{\ell,k}\left[\,(\bar{a}_\ell\, (\underline{x}_k + \overline{x}_k) - \delta_\ell(\mid \underline{x}_k \mid + \mid \overline{x}_k \mid)) + \right. \\
&\quad \left. sign(\bar{a}_\ell)\, (\bar{a}_\ell\, (\underline{x}_k - \overline{x}_k) - \delta_\ell(\mid \underline{x}_k \mid - \mid \overline{x}_k \mid))\,\right] \\
\overline{x}_{k+1} &= \frac{1}{2}\sum_{\ell=1}^{p}\mu_{i,k}\left[\,(\bar{a}_\ell\, (\underline{x}_k + \overline{x}_k) + \delta_\ell(\mid \underline{x}_k \mid + \mid \overline{x}_k \mid)) + \right. \\
&\quad \left. sign(\bar{a}_\ell)\, (\bar{a}_i\, (\overline{x}_k - \underline{x}_k) + \delta_\ell(\mid \overline{x}_k \mid - \mid \underline{x}_k \mid))\,\right]
\end{aligned}\right. \tag{16}$$

This latter formulation can be written under a matrix form as follows.

Proposition 3.1. *The state envelope of the scalar PLM* (7) *is given by:*

$$\begin{bmatrix} \underline{x}_{k+1} \\ \overline{x}_{k+1} \end{bmatrix} = \sum_{\ell=1}^{p}\mu_{\ell,k}\left(M_\ell \begin{bmatrix} \underline{x}_k \\ \overline{x}_k \end{bmatrix} + N_\ell \begin{bmatrix} \mid \underline{x}_k \mid \\ \mid \overline{x}_k \mid \end{bmatrix}\right) \tag{17}$$

with $s_\ell = sign(\bar{a}_\ell)$ *and*

$$M_\ell = \frac{1}{2}\begin{bmatrix} \bar{a}_\ell + \mid \bar{a}_\ell \mid & \bar{a}_\ell - \mid \bar{a}_\ell \mid \\ \bar{a}_\ell - \mid \bar{a}_\ell \mid & \bar{a}_\ell + \mid \bar{a}_\ell \mid \end{bmatrix} \qquad N_\ell = \frac{1}{2}\begin{bmatrix} -\delta_\ell - \delta_\ell\, s_\ell & -\delta_\ell + \delta_\ell\, s_\ell \\ \delta_\ell - \delta\ell\, s_\ell & \delta_\ell + \delta_\ell\, s_\ell \end{bmatrix}$$

The state equation (17) recursively provides the state envelope of a PLM with uncertain and bounded parameters from its initial condition $\left[\underline{x}_0\ \overline{x}_0\right]$.

3.2. The multidimensional case

Extending the previous section results from the scalar PLM (7) to the PLM of any order (1), the state envelope of the system (1) is given in the following proposition.

Proposition 3.2. *The state envelope of the PLM* (1) *is given by:*

$$\begin{bmatrix}\underline{x}_{k+1}\\ \overline{x}_{k+1}\end{bmatrix} = \sum_{\ell=1}^{p}\mu_{\ell,k}\left(M_\ell\begin{bmatrix}\underline{x}_{k}\\ \overline{x}_{k}\end{bmatrix} + N_\ell\begin{bmatrix}\mid\underline{x}_{k}\mid\\ \mid\overline{x}_{k}\mid\end{bmatrix} + E\,u_k\right) \tag{18}$$

with:

$$\begin{cases} M_\ell &= \dfrac{1}{2}\begin{bmatrix}\bar{A}_\ell + \mid \bar{A}_\ell\mid & \bar{A}_\ell - \mid \bar{A}_\ell\mid\\ \bar{A}_\ell - \mid \bar{A}_\ell\mid & \bar{A}_\ell + \mid \bar{A}_\ell\mid\end{bmatrix}\\ N_i &= \dfrac{1}{2}\begin{bmatrix}-\Delta_\ell - \Delta_\ell * S_\ell & -\Delta_\ell + \Delta_\ell * S_\ell\\ \Delta_\ell - \Delta_\ell * S_\ell & \Delta_\ell + \Delta_\ell * S_\ell\end{bmatrix}\\ S_\ell &= sign(\bar{A}_\ell)\\ E &= \begin{bmatrix}B^T & B^T\end{bmatrix}^T \end{cases} \tag{19}$$

Proof. The system envelope $[\underline{x}_k\ \overline{x}_k]$ is defined and recursively obtained from (18). It remains to show that this envelope is guaranteed to contain the actual state of the system (1). For this, the following property is proved by induction on k:

$$\mathcal{H}_k : \underline{x}_k \leq x_k \leq \overline{x}_k \tag{20}$$

By definition of an $\underline{x}_0$ and $\overline{x}_0$, it holds: $x_0 \in [\underline{x}_0, \quad \overline{x}_0]$, so the property is satisfied for $k = 0$. Assuming that for a given $k \in \mathbb{N}$ the property $\mathcal{H}_k$ is true, then let us show that $\underline{x}_{k+1} \leq x_{k+1}$ is true (the second inequality in (20) is proved similarly) implying that $\mathcal{H}_{k+1}$ is also true. Based on the system model (1), it follows that

$$\begin{aligned} x_{k+1} &= \sum_{\ell=1}^{p}\mu_{\ell,k}\,(\bar{A}_\ell + \Delta_{\ell,k})\,x_k + B\,u_k\\ y_k &= Cx_k \end{aligned} \tag{21}$$

From (18) and (21), the discrepency between the actual state and its lower bound is:

$$\tilde{\underline{x}}_{k+1} = x_{k+1} - \underline{x}_{k+1} \tag{22}$$

Let us detail the computation of the i^{th} component of the interval state vector $[\underline{x}_k\ \overline{x}_k]$ with respect to the lower bound of the system state (the calculation of its upper bound is similar). Denoting $\bar{a}_{\ell,i,j}$, $\delta_{\ell,i,j}$ and $b_{i,j}$ the respective elements of matrices $\bar{A}_\ell$, Δ_ℓ and B, from (18) one gets:

$$\begin{aligned} \underline{x}_{k+1,i} &= \frac{1}{2}\sum_{\ell=1}^{p}\mu_{\ell,k}\sum_{j=1}^{n}\Big[(\bar{a}_{\ell,i,j} + \mid\bar{a}_{\ell,i,j}\mid)\,\underline{x}_{k,j} + (\bar{a}_{\ell,i,j} - \mid\bar{a}_{\ell,i,j}\mid)\,\overline{x}_{k,j}\\ &- \delta_{\ell,i,j}\left(\mid\underline{x}_{k,j}\mid + \mid\overline{x}_{k,j}\mid\right) + sign(\bar{a}_{\ell,i,j})\delta_{\ell,i,j}\left(\mid\underline{x}_{k,j}\mid - \mid\overline{x}_{k,j}\mid\right) + 2b_{i,j}\,u_{k,j}\Big] \end{aligned} \tag{23}$$

where in this expression we only use the ith line of the state matrices M_ℓ and N_ℓ. This result will be used latter.

Using (23) and (21), the i^{th} component of $\tilde{\underline{x}}_{k+1}$ can be written as:

$$
\begin{aligned}
\tilde{\underline{x}}_{k+1,i} &= x_{k+1,i} - \underline{x}_{k+1,i} \\
&= \sum_{\ell=1}^{p} \sum_{j=1}^{n} \mu_{\ell,k} \sigma_{i,\ell,j}
\end{aligned}
\tag{24}
$$

where $\sigma_{i,\ell,j}$ is defined by:

$$
\begin{aligned}
\sigma_{i,\ell,j} = (\bar{a}_{\ell,i,j} + \delta_{\ell,k,i,j})\, x_{k,j} - \frac{1}{2} \Big[(\bar{a}_{\ell,i,j} + \mid \bar{a}_{\ell,i,j} \mid)\, \underline{x}_{k,j} + (\bar{a}_{\ell,i,j} - \mid \bar{a}_{\ell,i,j} \mid)\, \overline{x}_{k,j} \\
- \delta_{\ell,i,j} \left(\mid \underline{x}_{k,j} \mid + \mid \overline{x}_{k,j} \mid \right) + sign(\bar{a}_{\ell,i,j}) \delta_{\ell,i,j} \left(\mid \underline{x}_{k,j} \mid - \mid \overline{x}_{k,j} \mid \right) \Big]
\end{aligned}
\tag{25}
$$

and where $\delta_{\ell,k,i,j}$ denotes the (i,j) entry of the matrix $\Delta_{\ell,k}$.

According to Hypothesis 2, two situations may occur according to the sign of $\bar{a}_{\ell,i,j}$. When $\bar{a}_{\ell,i,j} > 0$, one obtains the following inequalities from (25) :

$$
\begin{aligned}
\sigma_{i,\ell,j} &= (\bar{a}_{\ell,i,j} + \delta_{\ell,k,i,j})\, x_{k,j} - \left(\bar{a}_{\ell,i,j}\, \underline{x}_{k,j} - \delta_{\ell,i,j} \mid \underline{x}_{k,j} \mid \right) \\
&\geq (\bar{a}_{\ell,i,j} + \delta_{\ell,k,i,j})\, \underline{x}_{k,j} - \left(\bar{a}_{\ell,i,j}\, \underline{x}_{k,j} - \delta_{\ell,i,j} \mid \underline{x}_{k,j} \mid \right) \\
&\geq \delta_{\ell,k,i,j}\, \underline{x}_{k,j} + \delta_{\ell,i,j} \mid \underline{x}_{k,j} \mid \\
&\geq 0
\end{aligned}
\tag{26}
$$

A similar result is obtained when $\bar{a}_{\ell,i,j} < 0$.

Finally, according to (24) and (26), $\tilde{\underline{x}}_{k+1,i}$ is a sum of non negative terms and thus $x_{k+1,i} \geq \underline{x}_{k+1,i}$, i.e. $x_{k+1} \geq \underline{x}_{k+1}$.Thus, the first inequality in $\mathcal{H}_{k+1}$ has been demonstrated and the induction principle ensures that it holds $k \in \mathbb{N}$, $x_k \geq \underline{x}_k$, $\forall k \in \mathbb{N}$. Similarly, the study of $\tilde{\overline{x}}_{k+1} = \overline{x}_{k+1} - x_{k+1}$ leads to establish that $x_k \leq \overline{x}_k$, $\forall k \in \mathbb{N}$. As a conclusion, the system (18) is a valid envelope of the system state of (1). □

Remark 2. *In the definition of the matrix M_ℓ in* (18)-(19)*, one can note that the matrices $\bar{A}_\ell + \mid \bar{A}_\ell \mid$ and $\bar{A}_\ell - \mid \bar{A}_\ell \mid$ can also be expressed with the positive and negative parts of $\overline{A}_\ell$.*

3.3. Example

The considered example is a system (1) with $n_u = 2$ inputs and $n = 3$ state variables, defined by the equations:

$$
\begin{aligned}
x_{1,k+1} = {} & 0.87083 x_{1,k} - 0.0583 x_{2,k} + 0.1 x_{3,k} + \\
& \tanh(\bar{u}_{1,k})(0.05417 x_{1,k} - 0.1083 x_{2,k} - 0.1 x_{3,k}) + 0.05 u_{1,k} \\
x_{2,k+1} = {} & -0.44167 x_{1,k} + 0.3083 x_{2,k} + 0.1 x_{3,k} + \\
& \tanh(1\bar{,}k)(-0.2083 x_{1,k} - 0.0583 x_{2,k} - 0.1 x_{3,k}) + 0.5 u_{2,k} \\
x_{3,k+1} = {} & 0.0166 x_{1,k} + 0.1 x_{2,k} - 0.2333 x_{3,k} + \\
& \tanh(1\bar{,}k)(-0.1166 x_{1,k} + 0.1 x_{2,k} - 0.1667 x_{3,k}) + 0.5 u_{1,k} + 0.8 u_{2,k} \\
\bar{u}_{1,k} = {} & \frac{u_{1,k} - 0.32}{0.075}
\end{aligned}
\tag{27}
$$

Let us denote:

$$\begin{array}{rcl} \tanh(\bar{u}_{1,k}) & = & \mu_{1,k} - \mu_{2,k} \\ \mu_{1,k} & = & \dfrac{\tanh(\bar{u}_{1,k}) + 1}{2} \\ \mu_{2,k} & = & \dfrac{1 - \tanh(\bar{u}_{1,k})}{2} \end{array} \tag{28}$$

where μ_1 and μ_2 satisfy the convexity property (2). Substituting (28) into (27), the system model can be written in a polytopic form defined by (1) with:

$$A_1 = \begin{bmatrix} 0.925 & -0.050 & 0 \\ -0.650 & 0.250 & 0 \\ -0.100 & 0 & -0.40 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.250 & -0.250 & 0 \\ -0.250 & 0.750 & 0 \\ 0 & -0.2 & 0.9 \end{bmatrix}$$

$$\Delta_1 = \begin{bmatrix} 0.01 & 0.015 & 0 \\ 0.02 & 0.02 & 0 \\ 0.01 & 0 & 0.01 \end{bmatrix} \qquad \Delta_2 = \begin{bmatrix} 0.01 & 0.015 & 0 \\ 0.02 & 0.02 & 0 \\ 0 & 0.01 & 0.01 \end{bmatrix} \qquad B = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$\mu_{1,k} = \left(1 + \tanh\left(\frac{u_{1,k} - 0.32}{0.075}\right)\right)/2 \qquad \mu_{2,k} = 1 - \mu_{1,k} \tag{29}$$

Figure 1 depicts the time evolutions of the two inputs u_1 (top left graph) and u_2 (top middle graph), the weighting function μ_1 (top right graph) and the three system states (bottom graphs). The lower and upper bounds of the state variables are depicted (in red dotted lines), as well as the state trajectory (in black solid lines) obtained from (1) for random bounded entries $\Delta_{1,k}$ and $\Delta_{2,k}$ obtained from uniform distributions

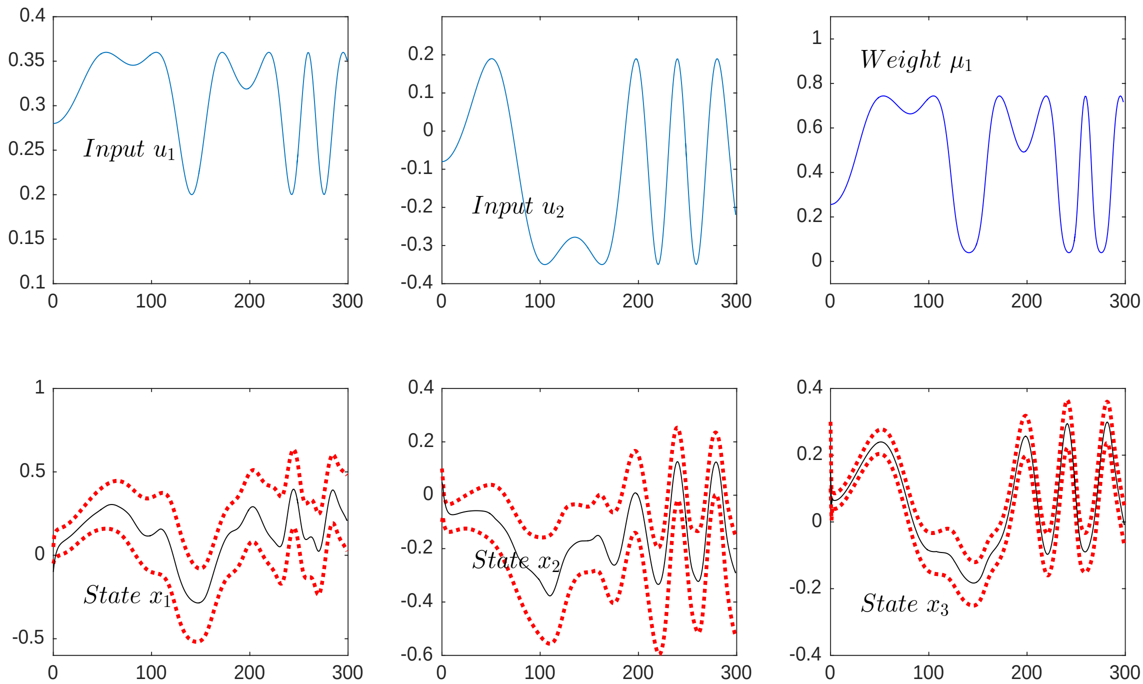

Figure 1. Inputs, weighting functions and state envelope of an uncertain PLM

whose respective amplitudes are between $-\Delta_1$ and Δ_1 and $-\Delta_2$ and Δ_2. It can be seen that the real state trajectory belongs to the determined envelope. Moreover, the time evolution of the weighting function μ_1 clearly illustrates the nonlinear nature of the PLM since the two submodels are time varyingly activated.

Remark 3. *If a stability analysis were desired, the absolute value terms in* (18) *might be tedious to deal with. In this case, similarly to what has been done in (Ichalal et al., 2018), the system* (18) *can be rewritten as a PLM without absolute value terms. Defining the generalized state* $z_k = \begin{bmatrix} \underline{x}_k^T & \overline{x}_k^T \end{bmatrix}^T$ *the system (18) takes the form:*

$$z_{k+1} \;=\; \sum_{\ell=1}^{p} \mu_{\ell,k} \left(M_\ell \, z_k + N_\ell \mid z_k \mid + E \, u_k \right) \tag{30}$$

where the matrices M_ℓ*,* N_ℓ *and* E *are defined in* (19)*.*

As detailed in (Tanaka & Wang, 2001), for any bounded function $f(w_k)$*, the nonlinear sector transformation allows to obtain a polytopic representation of* $f(w_k)w_k$*. Then, the component wise absolute value of a vector* w_k *can be written as* $\mid w_k \mid = f(w_k)w_k$*, where the bounded* $f(w_k)$ *is a diagonal matrix which diagonal entries are* 1 *or* -1 *depending on the sign of the components of* w_k*. In the scalar case, it gives:* $\mid w_k \mid = (\bar{\mu}_1(w_k) - \bar{\mu}_2(w_k))w_k$*, where* $\bar{\mu}_1$ *and* $\bar{\mu}_2$ *satisfy* (2) *and are defined by:* $\bar{\mu}_1(w_k) = \frac{sign(w_k)+1}{2}$ *and* $\bar{\mu}_2(w_k) = \frac{1-sign(w_k)}{2}$*. If* $dim(w_k) = 2$*, let us denote* $w_k = [w_{1,k} \; w_{2,k}]^T$ *and it holds:* $\mid w_k \mid = diag(\bar{\mu}_1(w_{1,k}) - \bar{\mu}_2(w_{1,k}), \bar{\mu}_1(w_{2,k}) - \bar{\mu}_2(w_{2,k}))w_k$*, which can also be written as:*

$$\mid w_k \mid = \sum_{j=1}^{4} \mu_j(w_k) \, E_j \, w_k \tag{31}$$

where the functions μ_j *satisfying* (2) *and the matrices* E_j *are given by:* $\mu_1(w_k) = \bar{\mu}_1(w_{1,k})\bar{\mu}_1(w_{2,k})$*,* $\mu_2(w_k) = \bar{\mu}_1(w_{1,k})\bar{\mu}_2(w_{2,k})$*,* $\mu_3(w_k) = \bar{\mu}_2(w_{1,k})\bar{\mu}_1(w_{2,k})$*,* $\mu_4(w_k) = \bar{\mu}_2(w_{1,k})\bar{\mu}_2(w_{2,k})$*,* $E_1 = I_2$*,* $E_2 = diag(1, -1)$*,* $E_3 = diag(-1, 1)$ *and* $E_4 = -I_2$*. From the definition of the functions* $\bar{\mu}_1$ *and* $\bar{\mu}_2$*, it follows that the functions* μ_j *readily satisfy the convex sum properties* (2)*. As detailed in (Ichalal et al., 2018), this polytopic expression of* $\mid w_k \mid$ *can be generalized to any vector* w_k *of any dimension and the number of terms in the sum* (31) *is* $r = 2^{2n}$ *.*

Finally, with (31)*, the system (30) is expressed under the following polytopic form:*

$$z_{k+1} = \sum_{\ell=1}^{p} \sum_{j=1}^{r} \mu_{\ell,k} \, \mu_j(z_k) \big((M_\ell + N_\ell \, E_j) z_k + E \, u_k \big) \tag{32}$$

where $r = 2^{2n}$ *and the functions* μ_j *and* $\mu_{\ell,k}$ *satisfy* (2) *and where no absolute value terms appear.*

4. Observer design for a system with bounded uncertain parameters

4.1. Observer structure

Before detailing the proposed observer, let us recall that the state envelope (18) may also be described by its lower and upper bounds:

$$\begin{cases} \underline{x}_{k+1} &= \sum_{\ell=1}^{p} \mu_{\ell,k} \frac{1}{2} \left[\left(\bar{A}_\ell + \mid \bar{A}_\ell \mid\right) \underline{x}_k + \left(\bar{A}_\ell - \mid \bar{A}_\ell \mid\right) \overline{x}_k \right. \\ & \left. - \left(\Delta_\ell + \Delta_\ell * S_\ell\right) \mid \underline{x}_k \mid - \left(\Delta_\ell - \Delta_\ell * S_\ell\right) \mid \overline{x}_k \mid \right] + B\, u_k \\ \overline{x}_{k+1} &= \sum_{\ell=1}^{p} \mu_{\ell,k} \frac{1}{2} \left[\left(\bar{A}_\ell - \mid \bar{A}_\ell \mid\right) \underline{x}_k + \left(\bar{A}_\ell + \mid \bar{A}_\ell \mid\right) \overline{x}_k \right. \\ & \left. + \left(\Delta_\ell - \Delta_\ell * S_\ell\right) \mid \underline{x}_k \mid + \left(\Delta_\ell + \Delta_\ell * S_\ell\right) \mid \overline{x}_k \mid \right] + B\, u_k \end{cases} \tag{33}$$

Given the structure (33), the observer design objective is to build a state estimate $\hat{x}_k$, based only on the input and output measurements of the system (21). The aim of the proposed interval observer (34) is to provide the envelope of the actual system state. Namely, it reduces to find the observer gains L_1 and L_2 in (34) such that $x_k \in [\hat{\underline{x}}_k\ \hat{\overline{x}}_k]$, for every instant k and to impose some dynamics to the state reconstruction error.

$$\begin{cases} \hat{\underline{x}}_{k+1} &= \sum_{\ell=1}^{p} \mu_{\ell,k} \frac{1}{2} \left[\left(\bar{A}_\ell + \mid \bar{A}_\ell \mid\right) \hat{\underline{x}}_k + \left(\bar{A}_\ell - \mid \bar{A}_\ell \mid\right) \hat{\overline{x}}_k + B\, u_k \right. \\ & \left. - \left(\Delta_\ell + \Delta_\ell * S_\ell\right) \mid \hat{\underline{x}}_k \mid - \left(\Delta_\ell - \Delta_\ell * S_\ell\right) \mid \hat{\overline{x}}_k \mid \right] + L_1(y_k - C\hat{\underline{x}}_k) \\ \hat{\overline{x}}_{k+1} &= \sum_{\ell=1}^{p} \mu_{\ell,k} \frac{1}{2} \left[\left(\bar{A}_\ell - \mid \bar{A}_\ell \mid\right) \hat{\underline{x}}_k + \left(\bar{A}_\ell + \mid \bar{A}_\ell \mid\right) \hat{\overline{x}}_k + B\, u_k \right. \\ & \left. + \left(\Delta_\ell - \Delta_\ell * S_\ell\right) \mid \hat{\underline{x}}_k \mid + \left(\Delta_\ell + \Delta_\ell * S_\ell\right) \mid \hat{\overline{x}}_k \mid \right] + L_2(y_k - C\hat{\overline{x}}_k) \end{cases} \tag{34}$$

Remark 4. *The observer structure in* (34) *is a proportional observer. As stated in Theorem 4.2, it guarantees the convergence of the state estimation error. Other structures may be latter studied, such as the proportional integral observer (PIO) offering better dynamical performances for the state reconstruction.*

4.2. Rationale for observer structure

The rationale for this structure is done in two steps. First, it must be shown that $\hat{\underline{x}}_k \leq x_k \leq \hat{\overline{x}}_k$. Secondly, the observer gains L_1 and L_2 must be adjusted to specify the temporal characteristics of the system state reconstruction error. Each step is addressed in Theorem 4.1 and Theorem 4.2 respectively.

Theorem 4.1. *The system (34) is an observer for (1), i.e. the state of the system (1) belongs to the envelope defined by (34).*

Proof. The proof proceeds by induction. Let $\mathcal{H}_k$ the property:

$$\mathcal{H}_k : \hat{\underline{x}}_k \leq x_k \leq \hat{\overline{x}}_k \tag{35}$$

The gap between the system state x_k and the lower bound $\hat{\underline{x}}_k$ of its observer, at time

k, is defined by:

$$\underline{e}_k = x_k - \hat{\underline{x}}_k \tag{36}$$

It is expressed at time $k+1$, from (1) and (34), by:

$$\left\{\begin{array}{rcl} \underline{e}_{k+1} & = & x_{k+1} - \hat{\underline{x}}_{k+1} \\ & = & \sum_{\ell=1}^{p} \mu_{\ell,k}(\bar{A}_\ell + \Delta_{\ell,k})\, x_k - L_1(y_k - C\,\hat{\underline{x}}_k) \\ & & -\frac{1}{2}\sum_{\ell=1}^{p} \mu_{\ell,k}\left[\left(\bar{A}_\ell + \mid \bar{A}_\ell \mid\right)\hat{\underline{x}}_k + \left(\bar{A}_\ell - \mid A_\ell \mid\right)\hat{\bar{x}}_k \right. \\ & & \left. -\,(\Delta_\ell + \Delta_\ell * S_\ell) \mid \hat{\underline{x}}_k \mid + (\Delta_\ell - \Delta_\ell * S_\ell) \mid \hat{\bar{x}}_k \mid\right] \end{array}\right. \tag{37}$$

The ith component of (37) is given by:

$$\left\{\begin{array}{rcl} \underline{e}_{k+1,i} & = & \sum_{\ell=1}^{p} \mu_{\ell,k} \sum_{j=1}^{n} \left((\bar{a}_{\ell,i,j} + \delta_{\ell,k,i,j})\, x_{k,j} - m_{1,i,j}(x_{k,j} - \hat{\underline{x}}_{k,j})\right) \\ & & -\frac{1}{2}\sum_{\ell=1}^{p} \mu_{\ell,k} \sum_{j=1}^{n} \left[(\bar{a}_{\ell,i,j} + \mid \bar{a}_{\ell,i,j} \mid)\, \hat{\underline{x}}_{k,j} + (\bar{a}_{\ell,i,j} - \mid \bar{a}_{\ell,i,j} \mid)\, \hat{\bar{x}}_{k,j} \right. \\ & & \left. -\,\delta_{i,j}\left(\mid \hat{\underline{x}}_{k,j} \mid + \mid \hat{\bar{x}}_{k,j} \mid\right) + sign(\bar{a}_{\ell,i,j})\delta_{i,j}\left(\mid \hat{\underline{x}}_{k,j} \mid - \mid \hat{\bar{x}}_{k,j} \mid\right)\right] \end{array}\right. \tag{38}$$

where $m_{1,i,j}$ are the coefficients of the $M_1 = L_1 C$ matrix. Similarly to what have been done previously, the ith component of this difference (38) can also be rewritten as:

$$\underline{e}_{k+1,i} = \sum_{\ell=1}^{p}\sum_{j=1}^{n} \mu_{\ell,k}\underline{e}_{k+1,i,j} \tag{39}$$

where the term denoted $\underline{e}_{k+1,i,j}$ is expressed according to the sign of the parameters $\bar{a}_{\ell,i,j}$.
If $\bar{a}_{\ell,i,j} > 0$, the term $\underline{e}_{k+1,i,j}$ is given by:

$$\begin{array}{rcl} \underline{e}_{k+1,i,j} & = & (\bar{a}_{\ell,i,j} + \delta_{k,i,j})\, x_{k,j} - \bar{a}_{\ell,i,j}\hat{\underline{x}}_{k,j} + \delta_{i,j} \mid \hat{\underline{x}}_{k,j} \mid - m_{1,i,j}(x_{k,j} - \hat{\underline{x}}_{k,j}) \\ & = & (\bar{a}_{\ell,i,j} + \delta_{k,i,j} - m_{1,i,j})\, x_{k,j} - \bar{a}_{\ell,i,j}\hat{\underline{x}}_{k,j} + \delta_{i,j} \mid \hat{\underline{x}}_{k,j} \mid + m_{1,i,j}\hat{\underline{x}}_{k,j} \\ & \geq & (\bar{a}_{\ell,i,j} + \delta_{k,i,j} - m_{1,i,j})\, \hat{\underline{x}}_{k,j} - \bar{a}_{\ell,i,j}\hat{\underline{x}}_{k,j} + \delta_{i,j} \mid \hat{\underline{x}}_{k,j} \mid + m_{1,i,j}\hat{\underline{x}}_{k,j} \\ & \geq & \delta_{k,i,j}\, \hat{\underline{x}}_{k,j} + \delta_{\ell,i,j} \mid \hat{\underline{x}}_{k,j} \mid \\ & \geq & 0 \end{array} \tag{40}$$

If $\bar{a}_{\ell,i,j} < 0$, the term $\underline{e}_{k+1,i,j}$ is given by:

$$\begin{array}{rcl} \underline{e}_{k+1,i,j} & = & (\bar{a}_{\ell,i,j} + \delta_{k,i,j})\, x_{k,j} - \bar{a}_{\ell,i,j}\hat{\bar{x}}_{k,j}\delta_{i,j} \mid \hat{\bar{x}}_{k,j} \mid - m_{1,i,j}(x_{k,j} - \hat{\underline{x}}_{k,j}) \\ & = & (\bar{a}_{\ell,i,j} + \delta_{k,i,j} - m_{1,i,j})\, x_{k,j} - \bar{a}_{\ell,i,j}\hat{\bar{x}}_{k,j} + \delta_{i,j} \mid \hat{\bar{x}}_{k,j} \mid + m_{1,i,j}\hat{\underline{x}}_{k,j} \\ & \geq & (\bar{a}_{\ell,i,j} + \delta_{k,i,j} - m_{1,i,j})\, \hat{\bar{x}}_{k,j} - \bar{a}_{\ell,i,j}\hat{\bar{x}}_{k,j} + \delta_{i,j} \mid \hat{\bar{x}}_{k,j} \mid + m_{1,i,j}\hat{\underline{x}}_{k,j} \\ & \geq & \delta_{k,i,j}\hat{\bar{x}}_{k,j} + \delta_{i,j} \mid \hat{\bar{x}}_{k,j} \mid \\ & \geq & 0 \end{array} \tag{41}$$

This shows that $\underline{e}_{k+1,i}$ in (39) is the sum of all positive terms and therefore $x_{k+1,i} \geq$

$\underline{e}_{k+1,i}$, i.e. $x_{k+1} \geq \underline{e}_{k+1}$. The first inequality of the property $\mathcal{H}_{k+1}$ has then been demonstrated and the induction principle ensures that for all $k > 0$, we have $x_k \geq \underline{\hat{x}}_k$. In the same way, it can be established that $x_k \leq \hat{\overline{x}}_k$ by analysing the gap :

$$
\begin{aligned}
\overline{e}_{k+1} &= \hat{\overline{x}}_{k+1} - x_{k+1} \\
&= \frac{1}{2}\sum_{\ell=1}^{p} \mu_{\ell,k}\big[(\bar{A}_\ell - \mid \bar{A}_\ell \mid)\underline{\hat{x}}_k + (\bar{A}_\ell + \mid \bar{A}_\ell \mid)\hat{\overline{x}}_k + (\Delta_\ell - \Delta_\ell * S_\ell) \mid \underline{\hat{x}}_k \mid \\
&\quad + (\Delta_\ell + \Delta_\ell * S_\ell) \mid \hat{\overline{x}}_k \mid + L_2(y_k - C\overline{x}_k) - (\bar{A}_\ell + \Delta_{\ell,k})\, x_k\big]
\end{aligned} \tag{42}
$$

As a conclusion, the system (34) is an observer of the system (1), in the sense that $\underline{\hat{x}}_k \leq x_k \leq \hat{\overline{x}}_k$, for any $k \in \mathbb{N}$. □

Theorem 4.2. *The observer (34) for system (1) is obtained by minimizing $\gamma > 0$ under the LMI constraints* (43), *for $\ell = 1, \ldots, p$, where $P \in \mathbb{R}^{2n\times 2n}$ is a symmetric positive definite matrix, K_1 and K_2 are matrices in $\mathbb{R}^{n\times p}$, G_1 and $G_2 \in \mathbb{R}^{n\times n}$ are invertible matrices and ε_ℓ are positive real scalars.*

$$
\begin{bmatrix}
I_{2n} - P & * & * & * \\
0 & \varepsilon_\ell \tilde{\Delta}_\ell \tilde{\Delta}_\ell^T - \gamma^2 I_{3n} & * & * \\
G^T A_{\ell,0} - KC_0 & G^T \tilde{B}_{\ell,0} & P - G^T - G & * \\
0 & 0 & \tilde{G} & -\varepsilon_\ell I_n
\end{bmatrix} \prec 0 \tag{43}
$$

The matrices G, $\tilde{G}$ and K are the block diagonal matrices defined by $G = diag(G_1, G_2)$, $\tilde{G} = [G_1 \;\; -G_2]$ and $K = diag(K_1, K_2)$. The other matrices are defined by:

$$
\begin{aligned}
\tilde{\Delta}_\ell^T &= [\Delta_\ell \; 0_n \; 0_n] & A_{\ell,0} &= \begin{bmatrix} H_{\ell,1} & -H_{\ell,2} \\ -H_{\ell,2} & H_{\ell,1} \end{bmatrix} \\
\tilde{B}_{\ell,0} &= \begin{bmatrix} 0 & H_{\ell,4} & H_{\ell,3} \\ 0 & H_{\ell,3} & H_{\ell,4} \end{bmatrix} & C_0 &= \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}
\end{aligned} \tag{44}
$$

with

$$
\begin{aligned}
H_{\ell,1} &= \frac{1}{2}(\bar{A}_\ell + \mid \bar{A}_\ell \mid) & H_{\ell,2} &= \frac{1}{2}(\bar{A}_\ell - \mid \bar{A}_\ell \mid) \\
H_{\ell,3} &= \frac{1}{2}(\Delta_\ell - \Delta_\ell * S_\ell) & H_{\ell,4} &= \frac{1}{2}(\Delta_\ell + \Delta_\ell * S_\ell)
\end{aligned} \tag{45}
$$

After solving the LMI (43), the observer gains are defined by:

$$
L = (G^T)^{-1} K \qquad L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \tag{46}
$$

Before giving the proof, some materials, borrowed from (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994), are given in the following three lemmas.

Lemma 4.3 (Schur complement)**.** *For any symmetric negative definite matrix C, the inequality $A - BC^{-1}B^T \prec 0$ is equivalent to*

$$
\begin{pmatrix} A & B \\ * & C \end{pmatrix} \prec 0 \tag{47}
$$

Lemma 4.4 (Congruence)**.** *For any full row rank matrix M, the matrix inequality $C \prec 0$ is equivalent to $M^T C M \prec 0$.*

Lemma 4.5. *For any matrix Δ_k satisfying $\Delta_k^T \Delta_k \prec I$ and any matrices M and N of appropriate dimensions, the following inequality holds $M\Delta_k N^T + N\Delta_k^T M \preceq +\varepsilon M M^T + \varepsilon^{-1} N N^T$ for any real positive scalar ε.*

Proof. With definitions (45), the state reconstruction error $\underline{e}_{k+1}$ (37) are rewritten:

$$\begin{array}{rcl} \underline{e}_{k+1} & = & \displaystyle\sum_{\ell=1}^{p} \mu_{\ell,k} \big[(\bar{A}_\ell + \Delta_{\ell,k})\, x_k - L_1(y_k - C\,\underline{\hat{x}}_k) \\ & & -H_{\ell,1}\underline{\hat{x}}_k - H_{\ell,2}\hat{\bar{x}}_k + H_{\ell,4} \mid \underline{\hat{x}}_k \mid + H_{\ell,3} \mid \hat{\bar{x}}_k \mid \big] \\ & = & \displaystyle\sum_{\ell=1}^{p} \mu_{\ell,k} [\Delta_{\ell,k}\, x_k + (H_{\ell,1} - L_1 C)\underline{e}_k - H_{\ell,2}\bar{e}_k + H_{\ell,3} \mid \underline{\hat{x}}_k \mid + H_{\ell,4} \mid \hat{\bar{x}}_k \mid] \end{array} \tag{48}$$

and similarly for $\bar{e}_{k+1}$:

$$\begin{array}{rcl} \bar{e}_{k+1} & = & \displaystyle\sum_{\ell=1}^{p} \mu_{\ell,k} \big[-(\bar{A}_\ell + \Delta_{\ell,k})\, x_k + L_2(y_k - C\,\hat{\bar{x}}_k) \\ & & + \big[H_{\ell,2}\underline{\hat{x}}_k + H_{\ell,1}\hat{\bar{x}}_k + H_{\ell,3} \mid \underline{\hat{x}}_k \mid + H_{\ell,4} \mid \hat{\bar{x}}_k \mid \big] \\ & = & \displaystyle\sum_{\ell=1}^{p} \mu_{\ell,k} [-\Delta_{\ell,k}\, x_k + (H_{\ell,1} - L_2 C)\bar{e}_k - H_{\ell,2}\bar{e}_k + H_{\ell,3} \mid \underline{\hat{x}}_k \mid + H_{\ell,4} \mid \hat{\bar{x}}_k \mid] \end{array} \tag{49}$$

The two deviations defined in (48) and (49) are gathered in the following augmented state estimation error: $e_k^T = [\underline{e}_k^T \;\; \bar{e}_k^T]$ and ruled by the following discrete time system:

$$e_{k+1} = \sum_{\ell=1}^{p} \mu_{\ell,k} \big[(A_{\ell,0} - L\,C_0)\, e_k + B_{\ell,k}\, \omega_k \big] \tag{50}$$

where the matrices $A_{\ell,0}$, C_0 and L are defined by (44) and (46) and where the input vector ω_k and $B_{\ell,0}$ are defined by:

$$B_{\ell,k} = \begin{bmatrix} \Delta_{\ell,k} & H_{\ell,4} & H_{\ell,3} \\ -\Delta_{\ell,k} & H_{\ell,3} & H_{\ell,4} \end{bmatrix} \qquad \omega_k = \begin{bmatrix} x_k \\ \mid \underline{\hat{x}}_k \mid \\ \mid \hat{\bar{x}}_k \mid \end{bmatrix} \tag{51}$$

As widely known, the $\mathcal{L}_2$-gain of the system (50) from its input w to its state e is bounded by $\gamma > 0$, if there exists a positive definite Lyapunov function $V(e_k)$ such that $f_\gamma(e_k, \omega_k) = V(e_{k+1}) - V(e_k) + e_k^T e_k - \gamma^2 \omega_k^T \omega_k < 0$ along the trajectory of the system (Boyd et al., 1994). Let us define a quadratic Lyapunov function $V(e_k) = e_k^T P e_k$ with $P \succ 0$. Then along the state trajectory of (50), the function $f_\gamma(e_k, \omega_k)$ is defined by $f_\gamma(e_k, \omega_k) = \tilde{e}_k^T \mathcal{M}_k \tilde{e}_k$, with $\tilde{e}_k^T = \begin{bmatrix} e_k^T & \omega_k^T \end{bmatrix}$ and

$$\mathcal{M}_k = \begin{bmatrix} I_{2n} - P + \tilde{A}_k^T P \tilde{A}_k & \tilde{A}_k^T P \tilde{B}_k \\ \tilde{B}_k^T P \tilde{A}_k & \tilde{B}_k^T P \tilde{B}_k - \gamma^2 I_m \end{bmatrix} \tag{52}$$

where the matrix $\tilde{A}_k$ is defined by $\tilde{A}_k = \sum_{\ell=1}^{p} \mu_{\ell,k}(A_{\ell,0} - LC_0)$ and $\tilde{B}_k = \sum_{\ell=1}^{p} \mu_{\ell,k} B_{\ell,k}$.

Let us now prove that if the LMI (43) is satisfied, then $V(e_{k+1}) - V(e_k) + e_k^T e_k - \gamma^2 \omega_k^T \omega_k < 0$, implying that the state estimation error is stable and that the $\mathcal{L}_2$-gain from w to e is minimized.

If (43) is satisfied, then using the Lemma 4.3, the following matrix inequality is also satisfied for $\ell = 1, \ldots, p$

$$\begin{bmatrix} I_{2n} - P & * & * \\ 0 & -\gamma^2 I_{3n} & * \\ G^T A_{\ell,0} - KC_0 & G^T \tilde{B}_{\ell,0} & P - G^T - G \end{bmatrix} + \varepsilon_\ell^{-1} \begin{pmatrix} 0 \\ 0 \\ \tilde{G}^T \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \tilde{G}^T \end{pmatrix}^T + \varepsilon_\ell \begin{pmatrix} 0 \\ \tilde{\Delta}_\ell \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{\Delta}_\ell \\ 0 \end{pmatrix}^T \prec 0 \tag{53}$$

From (2) it follows that $\Delta_{\ell,k}$ can be written as $\Delta_{\ell,k} = \tilde{\Delta}_{\ell,k} \Delta_\ell$ with $\tilde{\Delta}_{\ell,k}^T \tilde{\Delta}_{\ell,k} \preceq I_n$. Invoking Lemma 4.5, if (53) is satisfied, then the following matrix inequality holds:

$$\begin{bmatrix} I_{2n} - P & * & * \\ 0 & -\gamma^2 I_{3n} & * \\ G^T A_{\ell,0} - KC_0 & G^T \tilde{B}_{\ell,0} & P - G^T - G \end{bmatrix} + \mathcal{S}\left(\begin{pmatrix} 0 \\ 0 \\ \tilde{G}^T \end{pmatrix} \tilde{\Delta}_{\ell,k} \begin{pmatrix} 0 \\ \tilde{\Delta}_\ell \\ 0 \end{pmatrix}^T \right) \prec 0 \tag{54}$$

Using (44) and (51), and noticing that $G^T B_{\ell,k} = G^T \tilde{B}_{\ell,0} + \tilde{G}^T \tilde{\Delta}_{\ell,k} \tilde{\Delta}_\ell^T$, it follows that (54) is equivalent to $\tilde{\mathcal{M}}_{\ell,k} \prec 0$, where $\tilde{\mathcal{M}}_{\ell,k}$ is defined by

$$\tilde{\mathcal{M}}_{\ell,k} = \begin{bmatrix} I_{2n} - P & * & * \\ 0 & -\gamma^2 I_{3n} & * \\ G^T A_{\ell,0} - KC_0 & G^T B_{\ell,k} & P - G^T - G \end{bmatrix} \tag{55}$$

Since the function $\mu_{\ell,k}$ satisfy the property (2), $\tilde{\mathcal{M}}_{\ell,k} \prec 0$ implies $\sum_{\ell=1}^{p} \mu_{\ell,k} \tilde{\mathcal{M}}_{\ell,k} = \tilde{\mathcal{M}}_k \prec 0$, where $\tilde{\mathcal{M}}_k$ is defined by

$$\tilde{\mathcal{M}}_k = \begin{bmatrix} I_{2n} - P & * & * \\ 0 & -\gamma^2 I_{3n} & * \\ G^T \tilde{A}_k & G^T \tilde{B}_k & P - G^T - G \end{bmatrix} \tag{56}$$

Using Lemma 4.4, $\tilde{\mathcal{M}}_k \prec 0$ implies that $T_k^T \tilde{\mathcal{M}}_k T_k \prec 0$, for any full row rank T_k. If T_k is defined by

$$T_k = \begin{bmatrix} I_{2n} & 0 \\ 0 & I_{3n} \\ \tilde{A}_k & \tilde{B}_k \end{bmatrix} \tag{57}$$

it readily follows that $T_k^T \tilde{\mathcal{M}}_k T_k = \mathcal{M}_k$, where $\mathcal{M}_k$ is defined in (52). Summing up, if the LMI (43) are satisfied, then $\mathcal{M}_k \prec 0$ and $f_\gamma(e_k, \omega_k) = V(e_{k+1}) - V(e_k) + e_k^T e_k - \gamma^2 \omega_k^T \omega_k < 0$, which achieves the proof. □

Remark 5. *In Theorem 4.2, the matrix inequality is not an LMI in the $\mathcal{L}_2$ gain γ, since γ^2 appears in (43). Nevertheless, it can obviously be turned into a LMI problem by defining the LMI variable $\bar{\gamma} = \gamma^2$ and by minimizing $\bar{\gamma}$ under the LMI (43). Since the parameter γ reflects the attenuation of the term ω_k on the estimation error e_k, minimizing γ leads to a minimal interval width and allows better precision.*

Remark 6. *The presented result may be improved by introducing additional degree of freedom in the LMI optimization to reduce the conservatism implied by a quadratic Lyapunov function. For that purpose, a multiple Lyapunov function* $V_k = e_k^T \left(\sum_{i=1}^{r} \mu_i(z_k)P_i\right) e_k$ *may be used like in (Guerra & Vermeiren, 2004). Some other relaxation schemes may also be used to obtain relaxed sufficient LMI conditions. For instance, in (Tuan, Apkarian, Narikiyo, & Yamamoto, 2001) some appropriate factorization are used and in (Kruszewski, Wang, & Guerra, 2008) the decreasing of the Lyapunov function is sought between two non consecutive sample times.*

5. Extension to the case of uncertain polytopic systems with unmeasurable premise variables

In the present paper, the premise variables were assumed to be measurable, but using the sector nonlinear transformation to obtain a PLM from a nonlinear model may lead to PLM with premise variables depending on unmeasurable state variables.

One should note that, on the one hand, the approach proposed in the previous section is based on the minimization of the $\mathcal{L}_2$ gain from an uncertainty-like term ω_k to the state estimation error and, on the other hand, this uncertainty-like approach has already been used to deal with polytopic systems with unmeasurable premise variables (Ichalal, Marx, Ragot, & Maquin, 2010). Combining these two facts, it is be possible to propose an interval observer for uncertain polytopic systems with unmeasurable premise variables.

Even if this interesting but tedious problem is not fully solved, the present section presents a preliminary study for the design of an interval observer for uncertain PLM with unmeasurable premise variables defined by

$$\begin{cases} x_{k+1} &= A_k\, x_k + B\, u_k, \quad x_k \in \mathbb{R}^n, \quad u_k \in \mathbb{R}^{n_u} \\ A_k &= \sum_{\ell=1}^{p} \mu_{\ell,k} \left(\bar{A}_\ell + \Delta_{\ell,k}\right) \\ y_k &= C\, x_k, \quad y_k \in \mathbb{R}^{n_y} \end{cases} \tag{58}$$

where $\mu_{\ell,k}$ depends on x_k.

In order to take into account the dependence of the premise variables on the estimated state, the observer is slightly modified from (34) to become:

$$\begin{aligned} \hat{\underline{x}}_{k+1} = \sum_{\ell=1}^{p} \hat{\underline{\mu}}_{\ell,k} \frac{1}{2} \Big[\left(\bar{A}_\ell + \mid \bar{A}_\ell \mid\right) \hat{\underline{x}}_k + \left(\bar{A}_\ell - \mid \bar{A}_\ell \mid\right) \hat{\bar{x}}_k + B\, u_k \\ - \left(\Delta_\ell + \Delta_\ell * S_\ell\right) \mid \hat{\underline{x}}_k \mid - \left(\Delta_\ell - \Delta_\ell * S_\ell\right) \mid \hat{\bar{x}}_k \mid \Big] + L_1(y_k - C\hat{\underline{x}}_k) \end{aligned} \tag{59a}$$

$$\begin{aligned} \hat{\bar{x}}_{k+1} = \sum_{\ell=1}^{p} \hat{\bar{\mu}}_{\ell,k} \frac{1}{2} \Big[\left(\bar{A}_\ell - \mid \bar{A}_\ell \mid\right) \hat{\underline{x}}_k + \left(\bar{A}_\ell + \mid \bar{A}_\ell \mid\right) \hat{\bar{x}}_k + B\, u_k \\ + \left(\Delta_\ell - \Delta_\ell * S_\ell\right) \mid \hat{\underline{x}}_k \mid + \left(\Delta_\ell + \Delta_\ell * S_\ell\right) \mid \hat{\bar{x}}_k \mid \Big] + L_2(y_k - C\hat{\bar{x}}_k) \end{aligned} \tag{59b}$$

where $\hat{\underline{\mu}}_{\ell,k}$ and $\hat{\bar{\mu}}_{\ell,k}$ depend respectively on $\hat{\underline{x}}k$ and $\hat{\bar{x}}_k$. Clearly, the activation functions of the model and of the observer depend on different variables. Following (Ichalal et al., 2010), to facilitate the comparison of the states of the system and those of the observer, and thus to write the state estimation errors, the system model (58) can be exactly rewritten under the two following equivalent forms (60, 61):

$$\begin{cases} x_{k+1} &= \sum_{\ell=1}^{p} \hat{\underline{\mu}}_{\ell,k} \left(\bar{A}_\ell + \Delta_{\ell,k}\right) x_k + B\, u_k + \underline{\nu}_k \\ \underline{\nu}_k &= \sum_{\ell=1}^{p} (\mu_{\ell,k} - \hat{\underline{\mu}}_{\ell,k}) \left(\bar{A}_\ell + \Delta_{\ell,k}\right) x_k \end{cases} \tag{60}$$

$$\begin{cases} x_{k+1} &= \sum_{\ell=1}^{p} \hat{\overline{\mu}}_{\ell,k} \left(\bar{A}_\ell + \Delta_{\ell,k}\right) x_k + B\, u_k + \overline{\nu}_k \\ \overline{\nu}_k &= \sum_{\ell=1}^{p} (\mu_{\ell,k} - \hat{\overline{\mu}}_{\ell,k}) \left(\bar{A}_\ell + \Delta_{\ell,k}\right) x_k \end{cases} \tag{61}$$

In the previous equations, the perturbation-like terms $\underline{\nu}_k$ and $\overline{\nu}_k$ are respectively given by $\underline{\nu}_k = \sum_{\ell=1}^{p} \left(\mu_{\ell,k} - \hat{\underline{\mu}}_{\ell,k}\right) \left(\bar{A}_\ell + \Delta_{\ell,k}\right) x_k$ and $\overline{\nu}_k = \sum_{\ell=1}^{p} \left(\mu_{\ell,k} - \hat{\overline{\mu}}_{\ell,k}\right) \left(\bar{A}_\ell + \Delta_{\ell,k}\right) x_k$. With definitions (59a) and (60), the state reconstruction error $\underline{e}_{k+1}$ are rewritten:

$$\begin{aligned} \underline{e}_{k+1} &= \sum_{\ell=1}^{p} \hat{\underline{\mu}}_{\ell,k} \big[(\bar{A}_\ell + \Delta_{\ell,k})\, x_k - L_1(y_k - C\, \hat{\underline{x}}_k) \\ &\quad - H_{\ell,1} \hat{\underline{x}}_k - H_{\ell,2} \hat{\overline{x}}_k + H_{\ell,4} \mid \hat{\underline{x}}_k \mid + H_{\ell,3} \mid \hat{\overline{x}}_k \mid \big] - \underline{\nu}_k \\ &= \sum_{\ell=1}^{p} \hat{\underline{\mu}}_{\ell,k} [\Delta_{\ell,k}\, x_k + (H_{\ell,1} - L_1 C) \underline{e}_k - H_{\ell,2} \overline{e}_k + H_{\ell,3} \mid \hat{\underline{x}}_k \mid + H_{\ell,4} \mid \hat{\overline{x}}_k \mid \big] - \underline{\nu}_k \end{aligned} \tag{62}$$

and similarly, with (59b) and (61), $\overline{e}_{k+1}$ is defined by:

$$\begin{aligned} \overline{e}_{k+1} &= \sum_{\ell=1}^{p} \hat{\overline{\mu}}_{\ell,k} \big[- (\bar{A}_\ell + \Delta_{\ell,k})\, x_k + L_2(y_k - C\, \hat{\overline{x}}_k) \\ &\quad + \big[H_{\ell,2} \hat{\underline{x}}_k + H_{\ell,1} \hat{\overline{x}}_k + H_{\ell,3} \mid \hat{\underline{x}}_k \mid + H_{\ell,4} \mid \hat{\overline{x}}_k \mid \big] + \overline{\nu}_k \\ &= \sum_{\ell=1}^{p} \hat{\overline{\mu}}_{\ell,k} [-\Delta_{\ell,k}\, x_k + (H_{\ell,1} - L_2 C) \overline{e}_k - H_{\ell,2} \overline{e}_k + H_{\ell,3} \mid \hat{\underline{x}}_k \mid + H_{\ell,4} \mid \hat{\overline{x}}_k \mid \big] + \overline{\nu}_k \end{aligned} \tag{63}$$

Finally, the augmented state error $e_k^T = [\underline{e}_k^T \;\; \overline{e}_k^T]$ is generated by the equation:

$$e_{k+1} = \sum_{\ell=1}^{p} \hat{\underline{\mu}}_{\ell,k} \sum_{\ell=1}^{p} \hat{\overline{\mu}}_{\ell,k} \left[(A_{\ell,0} - L\, C_0)\, e_k + B_{\ell,k}\, \omega_k \right] \tag{64}$$

where the matrices $A_{\ell,0}$, C_0 and L are defined by (44) and (46) and where the input vector ω_k and $B_{\ell,0}$ are defined by:

$$B_{\ell,k} = \begin{bmatrix} \Delta_{\ell,k} & H_{\ell,4} & H_{\ell,3} & -I & 0 \\ -\Delta_{\ell,k} & H_{\ell,3} & H_{\ell,4} & 0 & I \end{bmatrix} \qquad \omega_k^T = \left[x_k^T \;\; \mid \hat{\underline{x}}_k \mid^T \;\; \mid \hat{\overline{x}}_k \mid^T \;\; \underline{\nu}_k^T \;\; \overline{\nu}_k^T \right] \tag{65}$$

The structure of this last equation is very close to the one of (51), therefore the

Theorem 4.2 can be easily be adapted for PLM with unmeasured premise variables by using (65) in the LMI conditions of the Theorem 4.2 leading to the computation of the observer gains.

6. Example

The example of Section 3.3 is continued with the same system matrices (27) and an output defined by $C = [1\ 0\ 0]$. The LMI conditions (43) are solved using the appropriate functions of MATLAB's YALMIP toolbox. The obtained values are the following (the values of ε_1 and ε_2 are given in Table 1).

$$
\begin{aligned}
P &= \begin{bmatrix} 24.946 & 2.444 & -3.765 & 2.059 & -3.029 & -2.058 \\ 2.444 & 15.023 & -9.182 & -3.029 & 0.14 & -2.775 \\ -3.765 & -9.182 & 68.412 & -2.058 & -2.775 & 22.288 \\ 2.059 & -3.029 & -2.058 & 24.946 & 2.444 & -3.765 \\ -3.029 & 0.14 & -2.775 & 2.444 & 15.023 & -9.182 \\ -2.058 & -2.775 & 22.288 & -3.765 & -9.182 & 68.412 \end{bmatrix} \\
G &= \begin{bmatrix} 22.228 & 3.383 & -4.417 & 0 & 0 & 0 \\ 3.383 & 14.88 & -13.953 & 0 & 0 & 0 \\ -4.417 & -13.953 & 91.043 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22.228 & 3.383 & -4.417 \\ 0 & 0 & 0 & 3.383 & 14.88 & -13.953 \\ 0 & 0 & 0 & -4.417 & -13.953 & 91.043 \end{bmatrix} \\
K &= \begin{bmatrix} 2.704 & 0 \\ -0.528 & 0 \\ -0.495 & 0 \\ 0 & 2.704 \\ 0 & -0.528 \\ 0 & -0.495 \end{bmatrix} \qquad L = \begin{bmatrix} 0.131 & 0 \\ -0.075 & 0 \\ -0.011 & 0 \\ 0 & 0.131 \\ 0 & -0.075 \\ 0 & -0.011 \end{bmatrix}
\end{aligned} \tag{66}
$$

One should note that because of the terms $tanh((u_{1,k} - 0.32)/0.075)$ multiplying the state variables in (27), the system is nonlinear and thus a state observer cannot be constructed with the design proposed in (Ichalal et al., 2018). Thus, the present paper clearly generalizes the results of (Ichalal et al., 2018) to a wider system class.

The Figure 1 depicts the evolution of the two inputs of the system and of the weighting function μ_1. The observer's performance is shown in Figure 2. The three upper graphs depict, for each state variable, the state envelope determined according to the results of the Section 3. The three lower graphs depict, in red dotted lines, the estimated upper and lower bounds of the system state, according to the results of the Section 4. On these graphs, the black solid line is a possible state trajectory for a given sequence of the parameter uncertainties. As it appears on the Figure 3, the interval width of the estimated states provided by the observer (red dotted lines) is clearly smaller than the interval width of the envelope (blue solid lines). It clearly illustrates the role of the observer proposed in Section 4 in the reconstruction of the system states. tThis is due to the fact that the observer takes into account the measurements of the system output.

According to the Remark 4, the influence of the parameter $\bar{\gamma}$ on the accuracy of the estimation is studied in the Table 1. It can be clearly seen that when $\bar{\gamma}$ is minimal

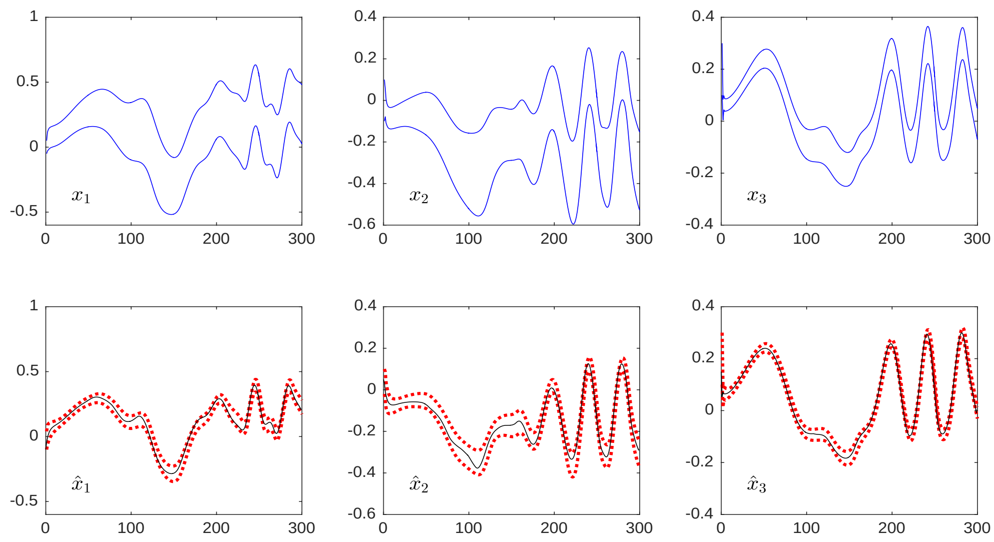

Figure 2. State envelope of the system (top) and state estimate provided by the observer (bottom)

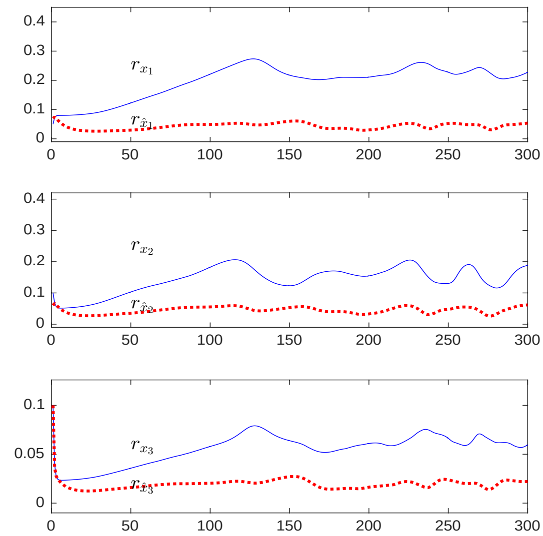

Figure 3. Widths of the system state envelope (r_{x_i} in blue) and of the estimated state bounds ($r_{\hat{x}_i}$ in red)

(here, when $\bar{\gamma} = 0.5$), the width of the estimated state bounds denoted $r_{\hat{x}_i}$ is smaller than when $\bar{\gamma}$ is not optimal (for instance when $\bar{\gamma} = 5$). In Table 1, the widths of the system envelope and of the estimated state bounds are averaged on 300 time samples. This short sensitivity analysis shows that the observer design has degrees of freedom that can be adjusted to an ad-hoc optimality criterion.

Table 1. Sensitivity of the radius of the intervals of the envelope and the estimate w.r.t. the parameter $\bar{\gamma}$

$\bar{\gamma}$	ε_1	ε_2	r_{x_1}	r_{x_2}	r_{x_3}	$r_{\hat{x}_1}$	$r_{\hat{x}_2}$	$r_{\hat{x}_3}$
0.5	24	21	0.1933	0.14	0.0533	0.0133	0.0233	0.01
5.0	56	43	0.1933	0.14	0.0533	0.0200	0.0266	0.0133

7. Conclusion

In this paper, the problem of state estimation for uncertain nonlinear system is addressed. Both nonlinearities and bounded uncertainties are taken into account with the polytopic formalism in order to have a representation of the system that can be used for state estimation. The state estimation is here understood in an interval setting. Put in other words, lower and upper bounds are sought such that, for every possible values of the uncertainties, the state variables lies between their respective lower and upper bounds. A first solution, the so-called state envelope, has been provided and proved to contain the state value. A more accurate solution has been brought by the proposed interval observer and the optimization of its gains has been studied.

The interest of this approach relies in the fact that it can easily be extended. On the one hand, the inclusion of uncertainties affecting all system matrices should be a direct extension. On the other hand, one can consider to change the observer structure to tackle the case of interval input and output measures. Finally, observer-based diagnosis of uncertain PLM may also be an interesting development of the proposed observer design.

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