Reducing Complexity of Nonlinear Dynamic Systems*

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Abstract. The complexity problem of nonlinear dynamic systems appears in a great number of scientific and engineering fields. The multi-model, also known as polytopic approach, constitutes an interesting tool for modeling dynamic nonlinear systems, in the framework of stability analysis and controller / observer design. A systematic procedure to transform a nonlinear system into a polytopic one will be briefly presented and illustrated by an academical example. This procedure gives the possibility of choosing between different polytopic structures, which is a degree of freedom used to ease the controllability, observability, stability analysis studies. In addition to that, the system transformation into polytopic form does not cause any information loss, contrarily to most existing studies in the field.

In the second part of this chapter, a discussion about multiple time scale nonlinear systems, also known as singularly perturbed systems is proposed, by eliminating some structural constraints and by performing the identification and the separation of the time-scales. Robust observer synthesis with respect to internal/external perturbations, modeling parametrization errors and unknown inputs are presented for the estimation of different variables of interest, the state variables.

The above-mentioned points will be applied to an activated sludge wastewater treatment plant (WWTP), which is a complex chemical and biological process. The variations in wastewater flow rate / composition and the time-varying bio-chemical reactions make this process nonlinear. Despite the process nonlinearity and complexity, there is a need to control the quality of the water rejected in the nature by the WWTPs in order to achieve the requirements of the European Union in terms of environmental protection. To this end, a Benchmark, proposed by the European program COST 624 to asses the control strategies of WWTPs, is used as an example in the present chapter.

1 Problem formulation

This chapter explains how the multi-model approach (also called polytopic [1] or Takagi-Sugeno approach [27]) can be used to model dynamical nonlinear systems, for observer /

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controller design or fault diagnosis purposes.

It is known that the dynamical and the nonlinear attributes are features of most existing technological or environmental processes. The modeling complexity problem is consequently an important element in a great number of scientific and engineering fields. When dealing with such systems, there is a necessity to develop systems operating over a wide range of functioning conditions and handle, in a most simple way, this complexity. Generally, a nonlinear system under the state space form is written as:

$$\dot{x}(t) = f(x(t), u(t)) \tag{1a}$$

$$y(t) = g(x(t), u(t)) \tag{1b}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the system state, $u(t) \in \mathbb{R}^{n_u}$ is the known input, $y(t) \in \mathbb{R}^{n_y}$ is the measured output and where f and g are nonlinear functions depending on the states x and inputs u. In real processes, these functions can be very complex and impossible to exploit for control and diagnosis purposes.

In the last decades, the multi-model approach seems to be an interesting tool to deal with complex nonlinear systems and thus has been intensively studied. It is the reason why it has been chosen for our recent research studies.

The basic principle of the multi-model approach is to replace an unique global model, as the one described by (1) -considered overly complex to be used as it is for different objectives, such as control, observer synthesis or fault diagnosis- by a set of simpler linear models defined as submodels. Roughly speaking, in the earlier works [27] each submodel describes the behavior of the considered process around a particular operating point and, thanks to a time varying interpolation mechanism between the different submodels, the global multi-model structure represents the original nonlinear model. Since the submodels are linear, it is an efficient way to address nonlinear problems by slightly adapting linear techniques [17].

The multi-model formalism is consequently based on time-varying interpolation between a set of linear sub-models. In the state space representation, the multi-model structure is presented as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z(t)) \left[A_i x(t) + B_i u(t)\right] \tag{2a}$$

$$y(t) = \sum_{i=1}^{r} \mu_i(z(t)) \left[C_i x(t) + D_i u(t)\right] \tag{2b}$$

where r is the number of submodels, the weighting functions $\mu_i(z(t))$ depend on the premise variables $z(t)$ and represent the weights of the submodels defined by the known matrices (A_i, B_i, C_i, D_i). The premise variables may depend on measurable signals (e.g. the system inputs or outputs) and / or on unmeasurable signals (e.g. the system state variables).

The functions $\mu_i(z(t))$ have the following properties:

$$\sum_{i=1}^{r} \mu_i(z(t)) = 1 \quad \text{and} \quad \mu_i(z(t)) \ \geq \ 0, \quad \forall t \in \mathbb{R}^+ \tag{3}$$

As mentioned in chapter 14 of [28], every nonlinear system can be written as a multi-model on a compact set of the state space, by using the so-called sector nonlinearity approach, that will be later on presented. The drawback of this technique is that no systematic choice of the premise variables has been realized. The choice of the premise variables plays a central role in the derivation of a the multi-model, since it impacts the structure of the submodels and thus on their use for performance analysis and observer / controller design. Other techniques to obtain a multi-model exist, such as linearization of the nonlinear model around one/several operating points, or dynamic linearization near arbitrary trajectory [16], system identification using experimental data [9]. Nevertheless, these different techniques are not general and systematical methodologies, depending, on one hand, on the choice of operating points (trajectory), and on the other hand, on the available data.

An analytic multi-modeling procedure with a motivated choice of the premise variable is presented in [18] and will be used as a nucleus point for modeling in this chapter. The proposed methodology avoids the inconveniences of the previously mentioned existing works: the choice of the linearization points is not necessary. and the transformation is realized without loss of information. Indeed, the obtained system has exactly the same dimension (simplification of MM systems by model order reduction is dealt in [14]) and state trajectory as the initial system. The complexity reduction comes from the fact that many analysis and / or design methods dedicated to linear systems have been extended to MM systems, and thus can be used to deal with nonlinear systems. The main points of these analytical rewriting technique will be described, illustrated and discussed in section 2.

2 Analytic procedure to obtain multi-model structure

This part is dedicated to the general methodology of transforming a given nonlinear model (1) into a multiple model. The transformation is realized without loss of information, the obtained system has exactly the same state trajectory as the initial system. The proposed method is analytical, and the obtained multi-model is equivalent to the initial nonlinear system.

Given a nonlinear system (1) with bounded nonlinearities, a multi-model state representation (2) can be obtained. This multi-model representation constitutes a linear parameter varying (LPV) system because the convex combinations of constant matrices calculated from the polytopes vertices give rise to matrices with variable parameters. The vertices are obtained using the convex polytopic transformation (CPT), given by the lemma 1. The constant matrices define the submodels and the nonlinearities are rejected into the submodel weighting functions. The multi-model obtained with this method is not unique: it depends on the choice of the lower and upper bounds of the nonlinearities used in the CPT and on the factorization used to rewrite the nonlinear system as an LPV model.

Lemma 1 *Convex Polytopic Transformation [29, 28]*

Let $h(z(t))$ be a bounded and continuous function from $[z_0, z_1]$ to $\mathbb{R}$, with $z_0, z_1 \in \mathbb{R}^q$ and $q = dim(z)$. Then, for all $h_1 \geq \max_z\{h(z)\}$ and $h_2 \leq \min_z\{h(z)\}$, there exist two nonnegative functions F_1 and F_2

$$F_1(z(t)) = \frac{h(z(t)) - h_2}{h_1 - h_2}$$

$$F_2(z(t)) = \frac{h_1 - h(z(t))}{h_1 - h_2}$$

such that:

$$F_1(z(t)) + F_2(z(t)) = 1$$
$$h(z(t)) = F_1(z(t)) \cdot h_1 + F_2(z(t)) \cdot h_2$$

Let us briefly give the important points of the general method to obtain a multi-model structure from a nonlinear formulation, and afterwards illustrate this method by an academical example.

2.1 Analytical method

Firstly, using a direct factorization of the state x and the input u, the system (1) is transformed into a quasi-linear parameter varying (quasi-LPV) form:

$$\dot{x}(t) = A(x(t), u(t))\, x(t) + B(x(t), u(t))\, u(t) \tag{4a}$$
$$y(t) = C(x(t), u(t))\, x(t) + D(x(t), u(t))\, u(t) \tag{4b}$$

This form is a state and control pseudo-affine representation.

Secondly, the nonlinear entries of the matrices A, B, C and/or D in the variables x and u are considered as "premise variables" and denoted $z_j(x, u)$ $(j = 1, ..., q)$. Several choices of these premise variables are possible due to the existence of different quasi-LPV forms (for details on the selection procedure see [17]). To each quasi-LPV form, a premise variable set corresponds.

Thirdly, a convex polytopic transformation is performed for all the premise variables $(j = 1, ..., q)$; thus the premise variables will be split into two parts, as follows:

$$z_j(x, u) = F_{j,1}(z_j(x, u))\, z_{j,1} + F_{j,2}(z_j(x, u))\, z_{j,2} \tag{5}$$

where the scalars $z_{j,1}$, $z_{j,2}$ are defined by

$$z_{j,1} = \max_{x,u} \{z_j(x, u)\} \tag{6a}$$
$$z_{j,2} = \min_{x,u} \{z_j(x, u)\} \tag{6b}$$

and where the partition functions $F_{j,1}(z_j)$, $F_{j,2}(z_j)$ involved in (5) are:

$$F_{j,1}(z_j(x, u)) = \frac{z_j(x, u) - z_{j,2}}{z_{j,1} - z_{j,2}} \tag{7a}$$
$$F_{j,2}(z_j(x, u)) = \frac{z_{j,1} - z_j(x, u)}{z_{j,1} - z_{j,2}} \tag{7b}$$

Remark 1 *For q premise variables, $r = 2^q$ submodels will be obtained.*

The two partitions will contribute to the construction of submodels and to the corresponding weighting functions. Then, the weighting functions are defined by some products of the original functions F_{j,σ_i^j}, according to:

$$\mu_i(x, u) = \prod_{j=1}^{q} F_{j,\sigma_i^j}(z_j(x, u)) \tag{8}$$

Considering definition (7a)-(7b), the reader should remark that these functions respect the conditions (3). In definition (8), the indexes σ_i^j ($i = 1, ..., 2^q$ and $j = 1, ..., q$) are equal to 1 or 2 and indicates which partition of the j^{th} premise variable ($F_{j,1}$ or $F_{j,2}$) is involved in the i^{th} submodel.
The constant matrices A_i ($i = 1, ..., 2^q$) are obtained by replacing the premise variables z_j in the matrix A with the scalars defined in (6a)-(6b):

$$A_i = A(z_{1,\sigma_i^1}, ..., z_{q,\sigma_i^q}) \tag{9}$$

The form (2) is obtained by similarly defining the matrices B_i, C_i and D_i.

The multi-model is consequently a convex combination of linear submodels, the nonlinearity of the initial system being transferred into the weighting functions related to each sub-model.

2.2 Academical example

Let us consider the following nonlinear system:

$$\dot{x}_1 = \cos(x_1)x_2 + x_1^3 u \tag{10a}$$

$$\dot{x}_2 = \frac{1}{\sqrt{x_2}}x_1 + x_1^2 x_2 \tag{10b}$$

Firstly, the system (10) can be represented in a quasi-LPV form:

$$\dot{x} = A(x,u)\ x + B(x,u)\ u$$

Several state- and control-affine quasi-LPV forms can be obtained: for the first state equation (10a), this separation is clear because of the product between the function $cos(x_1)$ and the second state variable x_2. For the second term, $x_1^3 u$, we can either affect the nonlinearity x_1^3 in the control matrix B (11), or distribute this nonlinearity among the state vector (x_1 component) and the state matrix A (12):

$$A(x) = \begin{bmatrix} 0 & \cos(x_1) \\ \frac{1}{\sqrt{x_2}} & x_1^2 \end{bmatrix} \qquad B(x) = \begin{bmatrix} x_1^3 \\ 0 \end{bmatrix} \tag{11}$$

$$A(x,u) = \begin{bmatrix} x_1^2 u & \cos(x_1) \\ \frac{1}{\sqrt{x_2}} & x_1^2 \end{bmatrix} \qquad B(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

For the second state equation (10b), at least two possible decompositions are observed. The most obvious decomposition is obtained by factorizing the two terms by x_1 and x_2 respectively (11). Another possibility is to factorize the right hand terms of (10b) only by x_1, reducing in this way the number of premise variables to three (13).

$$A(x) = \begin{bmatrix} 0 & \cos(x_1) \\ \frac{1}{\sqrt{x_2}} + x_1 x_2 & 0 \end{bmatrix} \qquad B(x) = \begin{bmatrix} x_1^3 \\ 0 \end{bmatrix} \tag{13}$$

In the following we only focus on the derivation of the MM form. The choice criteria between the possible forms (11), (12) and (13) will be discussed later.

Considering (13), the premise variables linked to the chosen quasi-LPV form are:

$$\begin{array}{l} z_1(x) = \cos(x_1) \\ z_2(x) = x_1^3 \\ z_3(x) = \frac{1}{\sqrt{x_2}} + x_1 x_2 \end{array} \tag{14}$$

Secondly, the convex polytopic transformation is applied, for each premise variable $z_j(x)$ $(j = 1, ..., 3)$ for $x_1 \in [-2\pi, 2\pi]$ and $x_2 \in [0.1, 12]$. Then, using lemma 1, each premise variable will be partitioned into two parts:

$$z_1(x) = F_{1,1}(z_1) \cdot z_{1,1} + F_{1,2}(z_1) \cdot z_{1,2} \tag{15a}$$

$$z_2(x) = F_{2,1}(z_2) \cdot z_{2,1} + F_{2,2}(z_2) \cdot z_{2,2} \tag{15b}$$

$$z_3(x) = F_{3,1}(z_3) \cdot z_{3,1} + F_{3,2}(z_3) \cdot z_{3,2} \tag{15c}$$

where $F_{j,1}(z_j(x))$ and $F_{j,2}(z_j(x))$ are defined using lemma 1. For example:

$$F_{1,1}(z_1(x)) = \frac{\cos(x_1) - z_{1,2}}{z_{1,1} - z_{1,2}} \tag{16}$$

and so on. The bounds $z_{j,1}$ and $z_{j,2}$ are chosen as in (6). The functions $F_{j,1}$ and $F_{j,2}$ respectively represent the first and the second partition of each premise variable. Let us note that $A(x)$ involves z_1 and z_3 as premise variables, while z_2 is involved in $B(x)$. Then, the matrices A and B will be evaluated at the vertices of the polytopes defined by the partitions of the premise variables involved in these matrices.
Applying the convex polytopic transformation (lemma 1) to z_1 (15a), it follows:

$$A(z_1, z_3) = \begin{bmatrix} 0 & z_1(x) \\ z_3(x) & 0 \end{bmatrix} = F_{1,1}(x) \begin{bmatrix} 0 & z_{1,1} \\ z_3(x) & 0 \end{bmatrix} + F_{1,2}(x) \begin{bmatrix} 0 & z_{1,2} \\ z_3(x) & 0 \end{bmatrix}$$

Applying the convex polytopic transformation (lemma 1) to z_3 (15c), it follows:

$$\begin{aligned} A(z_1, z_3) = {} & F_{1,1}F_{3,1}(x) \begin{bmatrix} 0 & z_{1,1} \\ z_{3,1} & 0 \end{bmatrix} + F_{1,2}F_{3,1}(x) \begin{bmatrix} 0 & z_{1,2} \\ z_{3,1} & 0 \end{bmatrix} \\ & + F_{1,1}F_{3,2}(x) \begin{bmatrix} 0 & z_{1,1} \\ z_{3,2} & 0 \end{bmatrix} + F_{1,2}F_{3,2}(x) \begin{bmatrix} 0 & z_{1,1} \\ z_{3,2} & 0 \end{bmatrix} \end{aligned}$$

As indicated in (2), the same weighting functions have to multiply the matrices A, B and C. In order to also include the partitions of the premise variable z_2, involved in $B(z_2)$ but not in $A(z_1, z_3)$, the matrix A is multiplied by $F_{2,1}(x) + F_{2,2}(x) = 1$:

$$\begin{aligned} A(z_1, z_3) = {} & F_{1,1}(x)F_{2,1}(x)F_{3,1}(x) \begin{bmatrix} 0 & z_{1,1} \\ z_{3,1} & 0 \end{bmatrix} + F_{1,2}(x)F_{2,1}(x)F_{3,1}(x) \begin{bmatrix} 0 & z_{1,2} \\ z_{3,1} & 0 \end{bmatrix} \\ & + F_{1,1}(x)F_{2,2}(x)F_{3,1}(x) \begin{bmatrix} 0 & z_{1,1} \\ z_{3,1} & 0 \end{bmatrix} + F_{1,2}(x)F_{2,2}(x)F_{3,1}(x) \begin{bmatrix} 0 & z_{1,2} \\ z_{3,1} & 0 \end{bmatrix} \\ & + F_{1,1}(x)F_{2,1}(x)F_{3,2}(x) \begin{bmatrix} 0 & z_{1,1} \\ z_{3,2} & 0 \end{bmatrix} + F_{1,2}(x)F_{2,1}(x)F_{3,2}(x) \begin{bmatrix} 0 & z_{1,2} \\ z_{3,2} & 0 \end{bmatrix} \\ & + F_{1,1}(x)F_{2,2}(x)F_{3,2}(x) \begin{bmatrix} 0 & z_{1,1} \\ z_{3,2} & 0 \end{bmatrix} + F_{1,2}(x)F_{2,2}(x)F_{3,2}(x) \begin{bmatrix} 0 & z_{1,2} \\ z_{3,2} & 0 \end{bmatrix} \end{aligned} \tag{17}$$

The same transformations are performed on the matrix $B(z_2)$:

$$\begin{aligned} B(z_2) &= \begin{bmatrix} z_2(x) \\ 0 \end{bmatrix} = \begin{bmatrix} z_{2,1}F_{2,1}(x) + z_{2,2}F_{2,2}(x) \\ 0 \end{bmatrix} \\ &= F_{2,1}(x) \begin{bmatrix} z_{2,1} \\ 0 \end{bmatrix} + F_{2,2}(x) \begin{bmatrix} z_{2,2} \\ 0 \end{bmatrix} \\ &= [F_{1,1}(x) + F_{1,2}(x)]\,[F_{3,1}(x) + F_{3,2}(x)] \left\{ F_{2,1}(x) \begin{bmatrix} z_{2,1} \\ 0 \end{bmatrix} + F_{2,2}(x) \begin{bmatrix} z_{2,2} \\ 0 \end{bmatrix} \right\} \end{aligned} \tag{18}$$

Finally, from (17) and (18) one obtains:

$$A(z_1, z_3) = \sum_{i=1}^{8} \mu_i(x) A_i, \qquad B(z_2) = \sum_{i=1}^{8} \mu_i(x) B_i \tag{19}$$

where $\mu_i(x)$ are combination of $F_{j,k}(x)$ ($j = 1, 2, 3$ and $k = 1, 2$) and where

$$\begin{aligned} &A_1 = A_3 = \begin{bmatrix} 0 & z_{1,1} \\ z_{3,1} & 0 \end{bmatrix} && A_2 = A_4 = \begin{bmatrix} 0 & z_{1,1} \\ z_{3,2} & 0 \end{bmatrix} \\ &A_5 = A_7 = \begin{bmatrix} 0 & z_{1,2} \\ z_{3,1} & 0 \end{bmatrix} && A_6 = A_8 = \begin{bmatrix} 0 & z_{1,2} \\ z_{3,2} & 0 \end{bmatrix} \\ &B_1 = B_2 = B_5 = B_6 = \begin{bmatrix} z_{2,1} \\ 0 \end{bmatrix} && B_3 = B_4 = B_7 = B_8 = \begin{bmatrix} z_{2,2} \\ 0 \end{bmatrix} \end{aligned} \tag{20}$$

After this example it is interesting to present a systematic way of constructing the matrices A_i and B_i. For example, to determine A_3 and B_3, the triplet $\sigma_3 = (1, 2, 1)$ is used. This triplet codes the variable partitions occurring in the 3^{rd} submodel and σ_3^k denotes the k^{th} value in the triplet σ_3. According to the expression of A_3 and B_3, these matrices may be denoted: $A_3 = A(z_{1,\sigma_3^1}, z_{3,\sigma_3^3})$ and $B_3 = B(z_{2,\sigma_3^2})$, where z_{1,σ_3^1}, z_{2,σ_3^2} and z_{3,σ_3^3} are the scalars defined in (6). In a more general way, A_i and B_i ($i = 1, ..., 8$) are denoted:

$$\begin{aligned} A_i &= A(z_{1,\sigma_i^1}, z_{3,\sigma_i^3}) \\ B_i &= B(z_{2,\sigma_i^2}) \end{aligned}$$

Those notations are consistent with (20). Associated to A_3 and B_3, the definition of (19) is obtained by using the triplet σ_3. Indeed :

$$\mu_3(x) = F_{1,\sigma_3^1}(x) F_{2,\sigma_3^2}(x) F_{3,\sigma_3^3}(x) \tag{21}$$

Each function defining a premise variable being partitioned into two functions, there are 2^3 submodels and 2^3 weighting functions. To each submodel i corresponds a triplet σ_i which codes the variable partitions occurring in it. After multiplying the functions representing these partitions, the weighting function $\mu_i(x)$ corresponding to the i^{th} submodel is obtained. To express the constant matrices A_i and B_i, characterizing each submodel i ($i = 1, ..., 8$), we use the quasi-LPV form (13) of the system (10), where $A(x, u)$ and $B(x, u)$ were defined in (13).

2.3 Choice Criteria for Quasi-LPV Form

Most of the existing results concerning performance analysis or observer / controller design for MM systems are based on the solution of linear matrix inequalities (LMI) obtained by using the Lyapunov method. Because of the convex sum property of the weighting functions, the LMI are only evaluated at the polytope vertices (A_i, B_i, C_i, D_i) and the weighting functions do not occur in the resolution of the LMIs [28, 29]. Only the matrices A_i, B_i, C_i and D_i are involved in the LMIs. Moreover, it should be highlighted that the LMI formulation generally results in sufficient conditions, since only the convex sum properties of the weighting functions are used. As a consequence, even if all the quasi-LPV models are exact equivalent rewritings of the original nonlinear system, the analysis or design results (obtained from an LMI procedure) may not be identical for all the possible MM forms that can be built from a given nonlinear system. That is why the choice of the premise variable set and the corresponding submodels is a critical point in the MM form derivation and it is essential to propose choice criteria for the MM structure [17, 18] in order to obtain the most suitable MM form and thus reduce the complexity of a nonlinear system.

First of all, in the framework of controller/observer design, the controllability/observability of the system under MM form should be ensured. A necessary -but not sufficient- condition for LMI-based designs is that all the submodels are controllable/observable, thus, the quasi- LPV forms producing submodels that are not controllable/observable must be eliminated. For instance, in the previous example, the form (12) with $B_i = 0$, is not suitable for controller design since all the submodels are uncontrollable.

The number of LMI constraints used for analysis and design is directly linked to the number of submodels: it is linear or polynomial in r [28]. Obviously, the larger this number of LMI constraints is, the less likely a solution to the LMI optimization exists. Also from a computational point of view, it is thus useful to chose the quasi-LPV form with a minimal r, that is to say with a minimum number of premise variables.

In addition to that, the observer/controller design for MM with premise variables depending on the state variables is a lot more complex than if the premise variables are known [8, 7, 19, 32]. As a consequence, MM form with premise variables depending on a minimal number of state variables is preferable.

2.4 Multiple time scale case

Real systems can have multiple time scale dynamics. In this case, the singular perturbation theory is often used to systematically identify the different time scales and to decompose the system dynamics according to them [23, 11]. Nevertheless, it is generally not trivial to model a process under the standard singularly perturbed form. One of the main tasks to realize is the identification and separation of the so-called slow and fast dynamics. In [6] this identification / separation is realized for a particular biological process by comparing its kinetic parameters. But this approach is dedicated to biological processes that are far from encompassing all nonlinear systems. So, more general methods to identify different time scales were proposed in [24]. These methods are based on the evaluation of the jacobian eigenvalues of the linearized system and will be used here.

After the separation of the multiple-time scale dynamics, the standard singularly perturbed form is obtained. In the limit case, when the singular perturbed parameter tends towards zero, a reduced form can be derived, with a dynamic part expressed by ordinary differential equations and a static part expressed by analytic equations, allowing to reduce the complexity of the model and simplifying its use, for control, estimation and diagnosis.

2.4.1 The singularly perturbed form.

The standard formulation of the singular perturbed systems with two-time scales and unknown inputs (UI) can be expressed as follows:

$$\begin{aligned}
\epsilon\dot{x}_f(t) &= f_f(x_s(t), x_f(t), u(t), d(t), \theta(t), \epsilon) && \text{(22a)}\\
\dot{x}_s(t) &= f_s(x_s(t), x_f(t), u(t), d(t), \theta(t), \epsilon) && \text{(22b)}\\
y(t) &= g(x(t), u(t), d(t)) && \text{(22c)}
\end{aligned}$$

where $x = [x_f,\ x_s]^T$, $x_s \in \mathbb{R}^{n_s}$ and $x_f \in \mathbb{R}^{n_f}$ are respectively the slow and fast state variables, $u \in \mathbb{R}^{n_u}$ the input vector, $d \in \mathbb{R}^{n_d}$ the unknown input vector, $\theta \in \mathbb{R}^{n_\theta}$ the modeling uncertainty, $y \in \mathbb{R}^{n_y}$ the output vector, $f_f \in \mathbb{R}^{n_f}$, $f_s \in \mathbb{R}^{n_s}$, $g \in \mathbb{R}^{n_y}$ and ϵ is a small and positive scalar, known as *singular perturbed parameter*.

Model uncertainty $\theta(t)$ generally refers to a difference between the model and the real system. It can be caused by imperfect knowledge or changes of the process or of its operating conditions. It can also be due to malfunctions acting on the process parameters. The unknown inputs $d(t)$ allow to model external disturbances or unmeasured inputs of the system. The nonlinear dynamic model with two time scales (22) takes these internal and external uncertainties into account.

In the limit case where $\epsilon \to 0$, the degree of the system (22) degenerates from $n_f + n_s$ to n_s, and the system is approximated by the following reduced system:

$$\begin{aligned}
\bar{E}\dot{x}(t) &= \begin{bmatrix} f_f(x_f(t), x_s(t), u(t), d(t), \theta(t), 0) \\ f_s(x_f(t), x_s(t), u(t), d(t), \theta(t), 0) \end{bmatrix}\\
&= f(x(t), u(t), d(t), \theta(t)) && \text{(23a)}\\
y(t) &= g(x(t), u(t), d(t)) && \text{(23b)}
\end{aligned}$$

with $\bar{E}$ defined by:

$$\bar{E} = \begin{bmatrix} 0_{n_f} & 0 \\ 0 & I_{n_s} \end{bmatrix} \tag{24}$$

In order to obtain the standard singularly perturbed form (23) from a classical nonlinear modeling (1), the identification and separation of slow and fast dynamics is the key point [6, 26]. The mathematical homotopy method for the linearized system, proposed by [30] and later improved by [24], is used to link each state variable with an eigenvalue. By comparing the eigenvalues, the biggest (respectively the smallest) ones will be associated with the slowest (respectively fastest) dynamics. Note that the linearized system is only used to identify the slow and fast dynamics, but not for the observer design. An equivalent MM representation will be used for this purpose, as presented in the previous section 2.1.

2.4.2 The homotopy method.

Based on the eigenvalue analysis of the linearized system, the homotopy method allows the identification and separation of the slow and fast dynamics [24].
Let us consider the linearization of the nonlinear system (1) around various equilibrium points (x_0, u_0):

$$\dot{x}(t) = A_0 x(t) + B_0 u(t) \tag{25}$$

where $A_0 = \dfrac{\partial f(x,u)}{\partial x}\big|_{(x_0,u_0)}$ and $B_0 = \dfrac{\partial f(x,u)}{\partial u}\big|_{(x_0,u_0)}$.
Ordering the eigenvalues of A_0 according to $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{n_x}$, the biggest (resp. smallest) eigenvalue corresponds to the slowest (resp. fastest) dynamic. The separation is performed by fixing a threshold τ, such that: $\lambda_1 \leq ... \leq \lambda_{n_f} << \tau \leq \lambda_{n_f+1} \leq ... \leq \lambda_n$.
If every eigenvalue can be connected to a given state variable, thus the dynamics of every state can be quantitatively estimated. The homotopy method requires to consider a system such that there exists an obvious relation between the eigenvalues and the states, as for example the diagonalized matrix of the jacobian matrix A_0. Further details on this method will be given in section 4 with the application to the WWTP.

2.4.3 Singular multi-models.

A singular MM can be derived from a nonlinear singular systems in a similar way than the MM has been obtained from the nonlinear systems (1) in section 2. The singularly perturbed systems presented under a MM form with unknown inputs and modeling uncertainty has the following form:

$$\bar{E}\,\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t), u(t))\,[A_i(\theta(t))x(t) + B_i(\theta(t))u(t) + E_i d(t)] \tag{26a}$$

$$y(t) = Cx(t) + Du(t) + Gd(t) \tag{26b}$$

where the weighting functions $\mu_i(x,u)$ depend on the unmeasurable state variables $x \in \mathbb{R}^{n_x}$ and on the input variables $u \in \mathbb{R}^{n_u}$. The variables $d \in \mathbb{R}^{n_d}$ are the unknown inputs, $\theta(t) \in \mathbb{R}^{n_\theta}$ the modeling uncertainty and $y \in \mathbb{R}^{n_y}$ the output variables. The matrices E_i, C, D and G are known real matrices and $A_i(\theta(t))$, $B_i(\theta(t))$ are time varying matrices. The matrix $\bar{E}$ can be a singular matrix (*i.e.* $rank(\bar{E}) \leq n_x$). The functions $\mu_i(x,u)$ respect the convexity conditions. Since in most practical situations, the sensor location and characteristics do not depend on the operating conditions, it is realistic to consider linear time invariant output equation (26b). This assumption is satisfied by the WWTP considered as an application in this chapter.

2.5 Modeling uncertainties as UI in MM.

In most studies [32, 33, 21] the modeling uncertainties are norm bounded and are expressed additively in the state matrix of the dynamic nonlinear model [21]. In this chapter, more general class of modeling uncertainties is assumed.

Let us consider that the uncertainties $\theta(t) = [\theta_1(t), \theta_2(t), \cdots \theta_{n_\theta}(t)]^T$ occur linearly in A_i and B_i (26):

$$A_i(\theta(t)) = A_{i,0} + \sum_{j \in I_A} \theta_j(t) A_{i,j}, \quad B_i(\theta(t)) = B_{i,0} + \sum_{j \in I_B} \theta_j(t) B_{i,j} \tag{27}$$

The components $\theta_j(t)$ of the vector $\theta(t)$ are time-varying parameters. The index set I_A, with $n_A = card(I_A)$, (resp. I_B, with $n_B = card(I_B)$) gathers the indexes of the components of the vector $\theta(t)$ that are involved in the matrices $A_i(\theta(t))$ (resp. $B_i(\theta(t))$). Obviously, these sets satisfy the following property: $I_A \cup I_B = I_\theta$, where $I_\theta = \{1, ..., n_\theta\}$. Moreover, $\theta_j^A(t)$, for $j = 1, \ldots, n_A$ (resp. $\theta_j^B(t)$, for $j = 1, \ldots, n_B$) denote the components of $\theta(t)$ involved in $A_i(\theta(t))$ (resp. $B_i(\theta(t))$). The matrices $A_{i,0}$, $B_{i,0}$, $A_{i,j}$ ($i = 1, \cdots, r$ and $j \in I_A$) and $B_{i,j}$ ($i = 1, \cdots, r$ and $j \in I_B$) are constants known matrices.

These uncertainties cause changes in the model parameters and may impact on the system stability. They are called multiplicative faults since they appear as product terms in (26). The main goal in the state estimation framework, is to minimize the influence of these parameter changes on the state estimation error. To this aim, these time-varying parameters can be considered as unknown inputs, by augmenting $d(t)$. Substituting the uncertain matrices (27) in (26) yields to:

$$\begin{aligned}
\bar{E}\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(z(t)) \left[\left(A_{i,0} + \sum_{j \in I_A} \theta_j(t) A_{i,j} \right) x(t) \right. \\
&\quad \left. + \left(B_{i,0} + \sum_{j \in I_B} \theta_j(t) B_{i,j} \right) u(t) + E_i d(t) \right] \\
y(t) &= Cx(t) + Du(t) + Gd(t)
\end{aligned} \tag{28}$$

Defining the augmented UI and its incidence matrices by:

$$\begin{aligned}
\bar{d}(t) &= \begin{bmatrix} (\theta_1^A(t)x(t))^T & \ldots & (\theta_{n_A}^A(t)x(t))^T & (\theta_1^B(t)u(t))^T & \ldots & (\theta_{n_B}^B(t)u(t))^T & d^T(t) \end{bmatrix}^T \\
\bar{F}_i &= \begin{bmatrix} A_{i,1} & \ldots & A_{i,n_A} & B_{i,1} & \ldots & B_{i,n_B} & E_i \end{bmatrix} \\
\bar{G} &= \begin{bmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 & G \end{bmatrix}
\end{aligned}$$

the uncertain MM with UI can be written as the following MM (with an augmented UI but no uncertain terms):

$$\begin{aligned}
\bar{E}\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x(t), u(t)) \left[A_{i,0}x(t) + B_{i,0}u(t) + \bar{F}_i \bar{d}(t) \right] \\
y(t) &= Cx(t) + Du(t) + \bar{G}\bar{d}(t)
\end{aligned} \tag{29}$$

3 Observer synthesis for singular multi-models

As seen in the previous section, the singular MM approach is a powerful tool to represent two time scale nonlinear systems, at least on a compact set of the state space (chapter 14 of [28],

[17]). It should be highlighted that although the CPT naturally leads to MM with premise variables depending on the state variable, and thus being unmeasurable, most of the existing works on MM consider measured premise variables. Only a few works are devoted to MM with unmeasurable premise variables (UPV) depending on the state variables [7, 19, 20, 3]. Since state estimation is known to be a crucial step in process control or diagnosis, thus observer design for singular MM affected by unknown input (UI) with UPV is of interest.

On the one hand, some works are devoted to state estimation of nonsingular MM with UPV [32, 2, 8], which is not trivial since the weighting functions used to synthesize the observer cannot depend on the state variables and will involve their estimates. On the other hand, many works deal with observer design for singular systems (see the book [31] and the references therein) and some of them are dedicated to state estimation of singular MM with UI [15], but the premise variables are supposed to be measured.

Here an unknown input observer (UIO) for descriptor MM with UPV is proposed. The proposed observer is a nonsingular MM in order to simplify the implementation. The existence conditions of the observer are expressed through linear matrix inequalities (LMI) by using the Lyapunov method and the $\mathcal{L}_2$ approach. The LMI approach has been chosen since it is well known to be a convenient tool to formulate various design objectives (stability, norm-bound, etc) [4]. In the following, the system under consideration is a singularly perturbed nonlinear system with two time scales (23), rewritten as a singular MM with UI and UPV(29).

Hypothesis 1 *The model* (29) *satisfies the following rank condition*

$$rank(W) = rank\left(\begin{bmatrix} W \\ Y \end{bmatrix}\right) \tag{30}$$

where, denoting the Kronecker product by $\otimes$*,* W *and* Y *are defined by:*

$$W = \left[\begin{array}{cc|ccc} \bar{E} & 0_{n_x \times n_d} & \bar{F}_1 & \cdots & \bar{F}_r \\ C & \bar{G} & 0_{n_y \times n_d} & \cdots & 0_{n_y \times n_d} \\ \hline 0_{rn_y \times n_x} & 0_{rn_y \times n_d} & & I_r \otimes \bar{G} & \end{array}\right] \tag{31a}$$

$$Y = \left[\begin{array}{cc|c} I_{n_x} & 0_{n_x \times n_d} & 0_{n_x \times rn_d} \end{array}\right] \tag{31b}$$

In order to simplify its implementation, the proposed following observer is chosen to be a nonsingular MM, even if the system to estimate is singular, :

$$\dot{\xi}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}, u) \left[N_i \xi(t) + G_i u(t) + L_i y(t)\right] \tag{32}$$

$$\hat{x}(t) = \xi(t) + T_2 y(t) - T_2 D u(t) \tag{33}$$

where $\hat{x}(t)$ denotes the state estimate. The state estimation error is given by

$$e(t) = x(t) - \hat{x}(t) \tag{34}$$

It is important to note that the weighting functions μ involved in the observer (32) depends on the $\hat{x}$ and thus the observer is nonlinear.

The observer design reduces to finding the gains N_i, G_i, L_i and T_2 such that the state estimation error obey to a stable generating system.

Theorem 1 *The observer* (32) *for the system* (26) *is obtained by finding a symmetric and positive definite matrix* $X \in \mathbb{R}^{n_x \times n_x}$ *and a matrix* $\tilde{Z} \in \mathbb{R}^{n_x \times (n_x + n_y(r+1))}$ *that minimize the positive scalar* $\bar{\gamma}$ *under the following LMI constraints:*

$$\begin{bmatrix} \Phi_i & (X\,Y\,W^+ + \tilde{Z}\,W^\perp)\,\Omega \\ \Omega^T\,(X\,Y\,W^+ + \tilde{Z}\,W^\perp)^T & -\bar{\gamma} I \end{bmatrix} < 0 \qquad i = 1, ..., r \tag{35}$$

where the matrices Ω *and* Φ_i *are defined by*

$$\begin{aligned} \Omega &= [\ I_n \quad 0 \ \mid\ 0 \quad \cdots \quad 0\ ]^T \\ \Phi_i &= (YW^+Y_i)^T X + X\,(Y\,W^+\,Y_i) + (W^\perp Y_i)^T \tilde{Z}^T + \tilde{Z}\,(W^\perp Y_i) + I \end{aligned} \tag{36}$$

with $W \in \mathbb{R}^{(n_x+n_y(r+1)) \times (n_x+n_d(r+1))}$ *and* $Y \in \mathbb{R}^{n_x \times (n_x+n_d(r+1))}$ *are defined by* (31), *and where* W^+ *is the pseudo inverse of* W, $W^\perp = I - WW^+$ *denotes the orthogonal of* W *verifying* $W^\perp W = 0$ *and where the matrices* $Y_i \in \mathbb{R}^{(n_x+n_y(r+1)) \times n_x}$ *are defined by*

$$Y_i = \left[\begin{array}{c} A_{i,0} \\ 0_{l \times n} \\ \hline v_i \otimes C \end{array} \right], \quad i = 1, ..., r \tag{37}$$

The vector $v_i \in \mathbb{R}^{r \times 1}$ *is the column vector containing* 1 *on the* i^{th} *entry and* 0 *on all the others.*
Once X *and* $\tilde{Z}$ *are obtained from LMI optimization* (35), *the matrices* Z, T_1, T_2 *and* K_i $(i = 1, ..., r)$ *can be deduced by*

$$Z = X^{-1}\,\tilde{Z} \tag{38}$$

$$\begin{bmatrix} T_1 & T_2 & \mid & K_1 & \ldots & K_r \end{bmatrix} = YW^+ + ZW^\perp \tag{39}$$

Finally, the observer gains are determined by

$$N_i = T_1 A_{i,0} + K_i C \tag{40}$$

$$G_i = T_1 B_{i,0} \tag{41}$$

$$L_i = N_i T_2 - K_i \tag{42}$$

Proof 1 *See [10]* □

4 Application to Wastewater Treatment Plant

4.1 Process description and ASM1 model

The widely used activated sludge wastewater treatment plant consists in mixing used waters with a rich mixture of bacteria in order to degrade the organic matter [22]. In this work, the data are generated by a part of the COST Benchmark [5]. The chosen WWTP configuration is a single tank (or bioreactor) and a settler (or clarifier), its general structure is depicted on figure 1. On figure 1, q_{in} represents the wastewater input flow, q_{out} the output flow, q_a the air flow and q_r (resp. q_w) are the recycled (resp. rejected) flow. The reactor volume V is assumed to be constant and thus the following equality is available for the reactor:

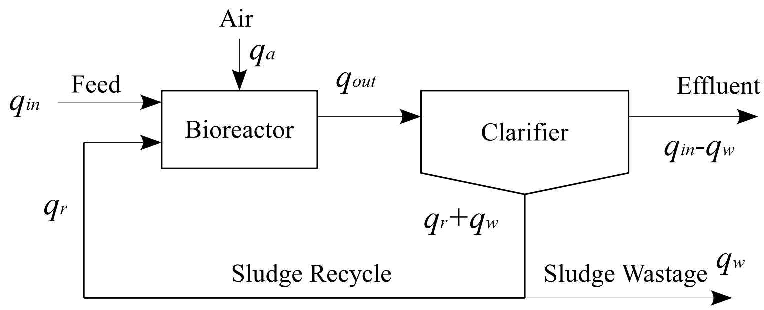

Figure 1: Wastewater treatment process diagram

$q_{out}(t) = q_{in}(t) + q_r(t)$. In general, $q_r(t)$ and $q_w(t)$ represent fractions of the input flow: $q_r(t) = f_r\, q_{in}(t)$, $1 \leq f_r \leq 2$, $q_w(t) = f_w\, q_{in}(t)$, $0 < f_w < 1$.

The polluted water circulates in the bioreactor where the bacterial biomass degrades the organic pollutant. Micro-organisms bring together in flocs and produce sludge, that is sent to the clarifier where the separation of the bacterial biomass from the purified water is made by gravity. A fraction of settled sludges is recycled towards the bioreactor to maintain its capacity of purification.

For observer/controller design, models of lower complexity are required since the full ASM1 model is quite complicated and may contain unnecessary informations for control and diagnosis tasks. Nevertheless, a quite complete model is considered here, since it involves the following components: soluble carbon S_S, particulate X_S, dissolved oxygen S_O, heterotrophic biomass X_{BH}, ammonia S_{NH}, nitrate S_{NO}, autotrophic biomass X_{BA}, soluble inert S_I, suspended inert X_I, soluble organic nitrogen S_{ND} and suspended organic nitrogen X_{ND}. Only the following components are not considered: the inert component X_P and the alkalinity S_{alk}. In practical situation, a single organic compound (denoted X_{DCO}) will be considered by adding the soluble part S_S and the particulate part X_S [25].

According to the setup of the European Benchmark COST 624 [5], it is assumed that $S_{O,in} \cong 0$, $S_{NO,in} \cong 0$ and $X_{BA,in} \cong 0$. Here, the operating conditions of the Bleesbrck (Luxembourg) WWTP are used for modeling and simulation: the concentrations $S_{NH,in}$, $X_{DCO,in}$ and $X_{BH,in}$ are not measured on line. Thus, $S_{NH,in}$ is considered as an unknown input and a daily mean value is used for $X_{DCO,in}$ and $X_{BH,in}$, which is a frequently used approximation. The measured concentrations are: the dissolved oxygen S_O, routinely measured in activated sludge WWTP, both nitrate S_{NO} and ammonia S_{NH} and the organic compound X_{DCO}. Consequently, the output $y = y(t)$, the input $u = u(t)$ and the unknown input $d = d(t)$ vectors are:

$$\begin{aligned} y &= [X_{DCO},\ \ S_O,\ \ S_{NH},\ \ S_{NO}]^T & (43)\\ u &= [X_{DCO,in},\ q_a,\ X_{BH,in}, S_{I,in},\ X_{I,in},\ S_{ND,in},\ X_{ND,in}]^T & (44)\\ d &= S_{NH,in} & (45) \end{aligned}$$

Let us consider the following explicit form of the ASM1:

$$
\begin{aligned}
\dot{X}_{DCO}(t) &= -\frac{1}{Y_h}[\varphi_1(t)+\varphi_2(t)] + (1-f_p)(\varphi_4(t)+\varphi_5(t)) + D_1(t) \\
\dot{S}_O(t) &= \frac{Y_h-1}{Y_h}\varphi_1(t) + \frac{Y_a-4.57}{Y_a}\varphi_3(t) + D_2(t) \\
\dot{S}_{NH}(t) &= -i_{xb}[\varphi_1(t)+\varphi_2(t)] - \left[i_{xb}+\frac{1}{Y_a}\right]\varphi_3(t) + D_3(t) \\
&\quad +(i_{xb}-f_p\, i_{xp})[\varphi_4(t)+\varphi_5(t)] \\
\dot{S}_{NO}(t) &= \frac{Y_h-1}{2.86Y_h}\varphi_2(t) + \frac{1}{Y_a}\varphi_3(t) + D_4(t) \\
\dot{X}_{BH}(t) &= \varphi_1(t)+\varphi_2(t)-\varphi_4(t)+D_5(t) \\
\dot{X}_{BA}(t) &= \varphi_3(t)-\varphi_5(t)+D_6(t) \\
\dot{S}_I(t) &= D_7(t) \\
\dot{X}_I(t) &= f_p[\varphi_4(t)+\varphi_5(t)]+D_8(t) \\
\dot{S}_{ND}(t) &= -\varphi_6(t)+\varphi_8(t)+D_9(t) \\
\dot{X}_{ND}(t) &= (i_{xb}-f_p i_{xp})[\varphi_4(t)+\varphi_5(t)]-\varphi_8(t)+D_{10}(t)
\end{aligned} \tag{46}
$$

where

$$
\begin{aligned}
\varphi_1(t) &= \mu_h \tfrac{X_{DCO}(t)}{K_{dco}+X_{DCO}(t)} \tfrac{S_O(t)}{K_{oh}+S_O(t)} X_{BH}(t) \\
\varphi_2(t) &= \mu_h \eta_{NOg} \tfrac{X_{DCO}(t)}{K_{dco}+X_{DCO}(t)} \tfrac{S_{NO}(t)}{K_{no}+S_{NO}(t)} \tfrac{K_{oh}}{K_{oh}+S_O(t)} X_{BH}(t) \\
\varphi_3(t) &= \mu_a \tfrac{S_{NH}(t)}{K_{nh,a}+S_{NH}(t)} \tfrac{S_O(t)}{K_{o,a}+S_O(t)} X_{BA}(t) \\
\varphi_4(t) &= b_h X_{BH}(t) \\
\varphi_5(t) &= b_a X_{BA}(t) \\
\varphi_6(t) &= k_a S_{ND}(t) X_{BH}(t) \\
\varphi_7(t) &= k_h \tfrac{X_{DCO}(t)/X_{BH}(t)}{K_{dco}+X_{DCO}(t)/X_{BH}(t)} \left(\tfrac{S_O(t)}{K_{oh}+S_O(t)} + \eta_h \tfrac{K_{oh}}{K_{oh}+S_O(t)} \tfrac{S_{NO}(t)}{K_{no}+S_{NO}(t)}\right) X_{BH}(t) \\
\varphi_8(t) &= k_h \tfrac{X_{ND}(t)/X_{BH}(t)}{K_{dco}+X_{DCO}(t)/X_{BH}(t)} \left(\tfrac{S_O(t)}{K_{oh}+S_O(t)} + \eta_h \tfrac{K_{oh}}{K_{oh}+S_O(t)} \tfrac{S_{NO}(t)}{K_{no}+S_{NO}(t)}\right) X_{BH}(t)
\end{aligned}
$$

and where $Y_a,\ Y_h,\ f_p,\ i_{xb},\ i_{xp}$ are constant coefficients and $K_{dco} = \frac{K_s}{f_{ss}}$.
The input/output balance is defined by:

$$
\begin{aligned}
D_1(t) &= D_{in}(t)\,[X_{DCO,in}(t) - X_{DCO}(t)] \\
D_2(t) &= D_{in}(t)\,[-S_O(t)] + Kq_a(t)\,[S_{O,sat} - S_O(t)] \\
D_3(t) &= D_{in}(t)\,[S_{NH,in}(t) - S_{NH}(t)] \\
D_4(t) &= D_{in}(t)\,[-S_{NO}(t)] \\
D_5(t) &= D_{in}(t)\left[X_{BH,in}(t) - X_{BH}(t) + \tfrac{f_r(1-f_w)}{f_r+f_w} X_{BH}(t)\right] \\
D_6(t) &= D_{in}(t)\left[-X_{BA}(t) + \tfrac{f_r(1-f_w)}{f_r+f_w} X_{BA}(t)\right] \\
D_7(t) &= D_{in}(t)[S_{I,in}(t) - S_I(t)] \\
D_8(t) &= D_{in}(t)\left[X_{I,in}(t) - X_I(t) + \tfrac{f_r(1-f_w)}{f_r+f_w}\right] X_I(t) \\
D_9(t) &= D_{in}(t)[S_{ND,in}(t) - S_{ND}(t)] \\
D_{10}(t) &= D_{in}(t)\left[X_{ND,in}(t) - X_{ND}(t) + \tfrac{f_r(1-f_w)}{f_r+f_w}\right] X_{ND}(t)
\end{aligned} \tag{47}
$$

where $D_{in}(t) = \frac{q_{in}(t)}{V}$. For numerical applications, the following heterotrophic growth and decay kinetic parameters are used [22]: $\mu_h = 3.733[1/24h]$, $\mu_a = 0.3[1/24h]$, $K_s = 20[g/m^3]$, $f_{ss} = 0.79$, $K_{oh} = 0.2[g/m^3]$, $K_{o,a} = 0.4[g/m^3]$, $K_{no} = 0.5[g/m^3]$, $K_{nh,a} = 1[g/m^3]$, $b_h = 0.3[1/24h]$, $b_a = 0.05[1/24h]$, $\eta_{NOg} = 0.8$. The stoichiometric parameters are $Y_h = 0.6$[g cell formed], $Y_a = 0.24$[g cell formed], $i_{xb} = 0.086$[g N in biomass], $i_{xp} = 0.06$[g N in endogenous mass], $f_p = 0.1$ and the oxygen saturation concentration is $S_{O,sat} = 10[g/m^3]$, $f_r = 1.1$ and $f_w = 0.04$, $V = 1333[m^3]$.

4.2 Slow and fast variable separation

In this section the identification of the slow and fast dynamics of the ASM1 model (46) is realized with the homotopy method [24], described in section 2.4. Let us consider the linearization of the nonlinear system (46) around various equilibrium points (x_0, u_0):

$$\dot{x}(t) = A_0 x(t) + B_0 u(t) \tag{48}$$

where $A_0 = \dfrac{\partial f(x,u)}{\partial x}\big|_{(x_0,u_0)}$ and $B_0 = \dfrac{\partial f(x,u)}{\partial u}\big|_{(x_0,u_0)}$.

For the considered model ASM1 (46), the separation of two time scale dynamics is confirmed by the eigenvalues of the jacobian A_0, depicted on figure 2 for forty operating points. Nine of the ten eigenvalues lie in $[-75 \;\; -1]$, while the last one is lower than -350. Setting a threshold at $\tau = -90$, it can be deduced that the system has one fast dynamic and nine slow dynamics:

$$x_F(t) = X_{DCO}(t) \tag{49}$$

$$x_S(t) = [\, S_O(t) \;\; S_{NH}(t) \;\; S_{NO}(t) \;\; X_{BH}(t) \;\; X_{BA}(t) \;\; S_I(t) \;\; X_I(t) \;\; S_{ND}(t) \;\; X_{ND}(t)\,]^T \tag{50}$$

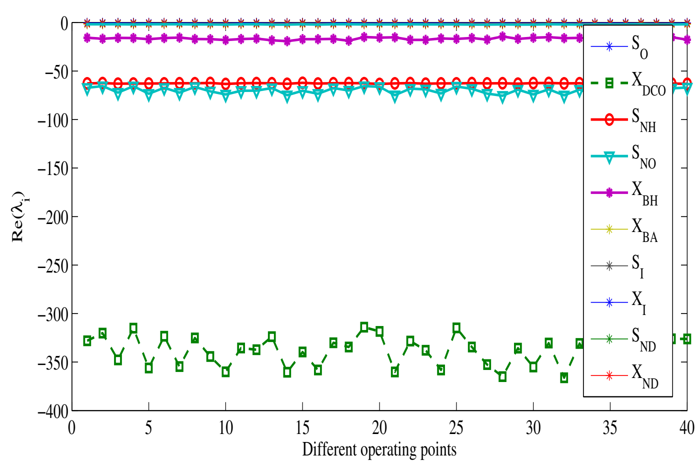

Figure 2: The eigenvalues of the linearized decoupled system

4.3 Singular multi-model representation for ASM1

The methodology proposed previously, in section 2, is applied here to obtain a multi-model structure for the ASM1 model. Based on the identification and the separation of the fast and slow dynamics, the obtained MM is singular.

Considering the process equations (46), it is natural to define the following premise variables since they mainly contribute to the definitions of the nonlinearity of the wastewater system:

$$\begin{aligned}
z_1(x,u) &= \frac{q_{in}(t)}{V} \\
z_2(x,u) &= \frac{X_{DCO}(t)}{K_{dco}+X_{DCO}(t)}\frac{S_O(t)}{K_{oh}+S_O(t)} \\
z_3(x,u) &= \frac{S_O(t)}{K_{o,a}+S_O(t)}\frac{S_{NH}(t)}{K_{nh,a}+S_{NH}(t)} \\
z_4(x,u) &= S_{ND}(t) \\
z_5(x,u) &= \frac{X_{DCO}(t)}{K_{dco}+X_{DCO}(t)}\frac{S_{NO}(t)}{K_{no}+S_{NO}(t)}\frac{K_{oh}}{K_{oh}+S_O(t)} \\
z_6(x,u) &= \frac{\frac{X_{ND}(t)}{X_{BH}(t)}}{K_{dco}+\frac{X_{DCO}(t)}{X_{BH}(t)}}\left[\frac{S_O(t)}{K_{oh}+S_O(t)}+\frac{\eta_h\, K_{oh}}{K_{oh}+S_O(t)}\frac{S_{NO}(t)}{K_{no}+S_{NO}(t)}\right]
\end{aligned} \tag{51}$$

According to the remark 1, 6 premise variables will result in 64 submodels, which can lead to infeasible LMI condition for the observer design. An alternative is to reduce the number of premise variables by considering some of them as constant terms equal to their mean value in the operating time interval. Figure 3 illustrates the evolution of the premise variables

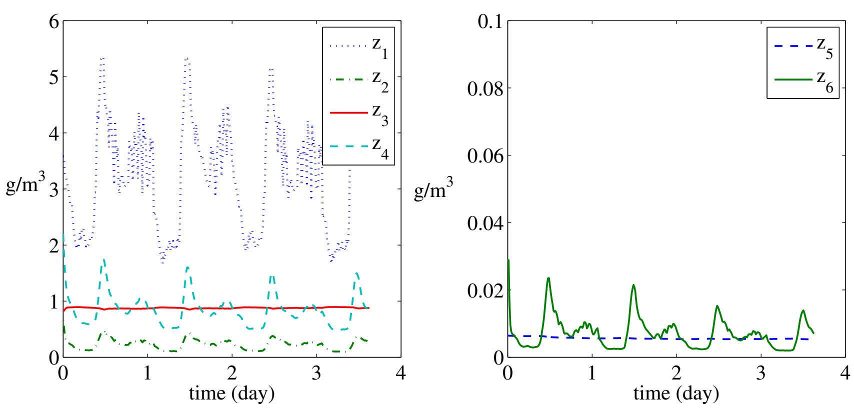

Figure 3: Evolution of the premise variables $z_1(t), \dots z_6(t)$

(51). The small variation ranges of the premise variables z_3, z_5 and z_6, compared to the others, encourage to consider their means value (respectively denoted $\tilde{z}_3$, $\tilde{z}_5$ and $\tilde{z}_6$) in the MM construction. Using this approximation, only three premise variables, namely z_1, z_2 and z_4, are considered to design the multi-model, which is thus described by 2^3 submodels.

The system (46) can be written in a following Quasi-LPV form with unknown input $\dot{x}(t) = A(x,u)x(t) + B(x,u)u(t) + F(x,u)d(t)$, where the matrices $A(x,u)$, $B(x,u)$ and $F(x,u)$, depending on the premise variables previously defined, are given by:

$$A(x,u) = \begin{bmatrix} a_{1,1} & 0 & 0 & 0 & a_{1,5} & a_{1,6} & 0 & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 & a_{2,5} & a_{2,6} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{3,3} & 0 & a_{3,5} & a_{3,6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{4,4} & a_{4,5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{6,6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{7,7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{8,5} & a_{8,6} & 0 & a_{8,8} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{9,5} & 0 & 0 & 0 & a_{9,9} & 0 \\ 0 & 0 & 0 & 0 & a_{10,5} & a_{10,6} & 0 & 0 & 0 & a_{10,10} \end{bmatrix}$$

$$B(u) = \begin{bmatrix} z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & K\,S_{O,sat} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_1 \end{bmatrix}, \; F(u) = \begin{bmatrix} 0 \\ 0 \\ z_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (52)$$

where $z_1 = z_1(u)$, the matrices components $a_{1,1}(x,u) = a_{3,3}(x,u) = a_{4,4}(x,u) = a_{7,7}(x,u) = a_{9,9}(x,u) = -z_1(u)$ and where:

$$\begin{aligned}
a_{1,5}(x,u) &= -\frac{\mu_h}{Y_h} z_2(x,u) + (1-f_p)\,b_h - \frac{\mu_h\,\eta_{NOg}}{Y_h}\tilde{z_5} \\
a_{1,6}(x,u) &= (1-f_p)\,b_a \\
a_{2,2}(x,u) &= -z_1(u) - K\,q_a \\
a_{2,5}(x,u) &= \frac{(Y_h-1)\mu_h}{Y_h} z_2(x,u) \\
a_{2,6}(x,u) &= -\frac{4.57 - Y_a}{Y_a}\mu_a\,\tilde{z_3} \\
a_{3,5}(x,u) &= (i_{xb} - f_p\,i_{xp})b_h - i_{xb}\,\mu_h\,z_2(x,u) - i_{xb}\,\mu_h\,\eta_{NOg}\,\tilde{z_5} \\
a_{3,6}(x,u) &= (i_{xb} - f_p\,i_{xp})\,b_a - (i_{xb} + \frac{1}{Y_a})\mu_a\,\tilde{z_3} \\
a_{4,5}(x,u) &= \frac{Y_h - 1}{2.86\,Y_h}\mu_h\,\eta_{NOg}\,\tilde{z_5} \\
a_{4,6}(x,u) &= \frac{1}{Y_a}\mu_a\tilde{z_3} \\
a_{5,5}(x,u) &= \mu_h\,z_2(x,u) - b_h + z_1(u)\left[\frac{f_w(1+f_r)}{f_r+f_w} - 1\right] + \mu_h\,\eta_{NOg}\,\tilde{z_5}
\end{aligned}$$

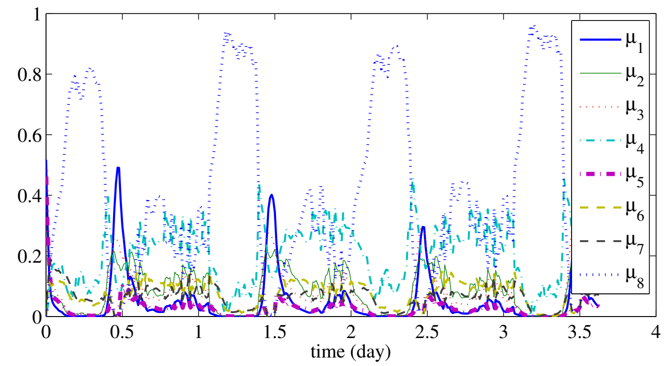

Figure 4: Weighting functions $\mu_i(z(x,u))$

$$
\begin{aligned}
a_{6,6}(x,u) &= z_1(u)\left[\frac{f_w(1+f_r)}{f_r+f_w}-1\right]-b_a\mu_a\,\tilde{z_3}\\
a_{8,5}(x,u) &= f_p b_h\\
a_{8,6}(x,u) &= f_p b_a\\
a_{8,8}(x,u) &= \left[\frac{f_r(1-f_w)}{f_r+f_w}-1\right]z_1(u)\\
a_{9,5}(x,u) &= -k_a\,z_4(x,u)+k_h\,\tilde{z_6}\\
a_{10,5}(x,u) &= (i_{xb}-f_p\,i_{xp})b_h-k_h\,\tilde{z_6}\\
a_{10,6}(x,u) &= (i_{xb}-f_p\,i_{xp})b_a\\
a_{10,10}(x,u) &= \left[\frac{f_r(1-f_w)}{f_r+f_w}-1\right]z_1(u) \qquad (53)
\end{aligned}
$$

The decomposition of the three premise variables -$z_1(u)$, $z_2(x,u)$ and $z_4(x,u)$- from (51) is realized by using the convex polytopic transformation (5). The scalars $z_{j,1}$ and $z_{j,2}$ are defined as in (6a)-(6b) and the functions $F_{j,1}(z_j(x,u))$ and $F_{j,2}(z_j(x,u))$ are given by (7a)-(7b) for $j=1,2,4$. By multiplying the functions $F_{j,\sigma_i^j}(z_j(x,u))$, the $r=8$ weighting functions $\mu_i(z(x,u))$ $(i=1,\cdots,8)$ are obtained and illustrated in figure 4:

$$\mu_i(z(x,u)) = F_{1,\sigma_i^1}(z_1(u))F_{2,\sigma_i^2}(z_2(x,u))F_{4,\sigma_i^4}(z_4(x,u)) \qquad (54)$$

The constant matrices A_i, B_i and F_i defining the 8 submodels, are determined by using the matrices $A(x,u)$, $B(u)$, $F(u)$ and the scalars z_{j,σ_i^j}:

$$
\begin{aligned}
A_i &= A(z_{1,\sigma_i^1}, z_{2,\sigma_i^2}, z_{4,\sigma_i^4}) && (55a)\\
B_i &= B(z_{1,\sigma_i^1}) && (55b)\\
F_i &= F(z_{1,\sigma_i^1}) \qquad i=1,...,8,\ j=1,2,4 && (55c)
\end{aligned}
$$

According to the fast and slow variable separation (49), performed in section 4.2, the matrix $\bar{E}$ of the singular multi-model formulation (23) is defined by:

$$\bar{E} = \textbf{diag}\begin{bmatrix}0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix} \qquad (56)$$

Thus, the nonlinear model (43)-(46) can be written as the following singular MM:

$$\begin{aligned}
\bar{E}\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x,u)[A_i x(t) + B_i u(t) + F_i d(t)] && (57a)\\
y(t) &= Cx(t) + Du(t) + Gd(t) + \delta(t) && (57b)
\end{aligned}$$

where D and G are null matrices of appropriate dimensions, C is defined by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{58}$$

and where $\delta(t)$ is a measurement noise modeled by a zero mean random signal.

In order to highlight and quantify the accuracy of the approximation of the ASM1 (46), provided by the singular MM (57), the average relative deviation (ARD) is computed for each state variable by:

$$ARD_j = \frac{1}{n_t} \sum_{i=1}^{n_t} \left(\frac{\left| x_{MM}^j(i) - x_{ASM1}^j(i) \right|}{x_{MM}^j(i)} \right) \times 100\%, \quad \text{for } j = 1, ..., n_x \tag{59}$$

where n_t is the number of data points. The following obtained values of the ARD:

$$ARD = [1.85 \quad 0.71 \quad 0.28 \quad 5.60 \quad 1.37 \quad 0.25 \quad 0.05 \quad 0.07 \quad 2.31 \quad 0.45]\,\%$$

confirm that the state trajectories of the original system (46) and of the approximated one (57) are close. In conclusion, the ASM1 model (46) can be rewritten under the singularly MM with unmeasurable premises, as described in (26) and the state estimation, proposed in section 3, can be applied.

4.4 Unknown input observer design

As seen is section 3, a nonsingular multi-observer (32) can be designed based on the singularly perturbed multiple model (26) or (29). The matrices $\bar{E}$, A_i, B_i, $\bar{F}_i$, C and $\bar{G}$ of (29) are defined by (55,56,58) and the weighting functions are defined in (54).

Let us consider the model uncertainties $\theta_1^A(t) = \theta_1(t)$, $\theta_2^A(t) = \theta_2(t)$ and $\theta_3^A = \theta_3(t)$ caused by the deviation of three model parameters from their following nominal values: $f_{ss} = 0.79$, $\eta_{NOg} = 0.8$ and $K_{no} = 0.5$ involved in the ASM1 model (46) (see figure 5). The uncertain parameter f_{ss} influences the dynamic of the states X_{DCO}, S_O, S_{NH}, X_{BH}, S_{ND} and X_{ND}. The parameter η_{NOg} interferes with the dynamic of the states X_{DCO}, S_{NH}, S_{NO}, X_{BH}. The uncertain parameter K_{no} affects the dynamics of X_{DCO}, S_{NH}, S_{NO}, S_{ND} and X_{ND}. Applying the theorem 1, the observer matrices N_i, G_i, L_i and T_2 are deduced, by using the specific solver Yalmip for convex optimization problems [12, 13].

In figure 6 the state variables and their estimates are presented. Figures 7 and 8 represent the unknown input $d(t)$ and the known input $u(t)$, respectively. The $\mathcal{L}_2$ gain of the transfer from $\omega(t)$ to $e(t)$ is bounded by $\gamma = 4.5$. The reconstructed output $\hat{y}(t) = C\,\hat{x}(t)$ of the

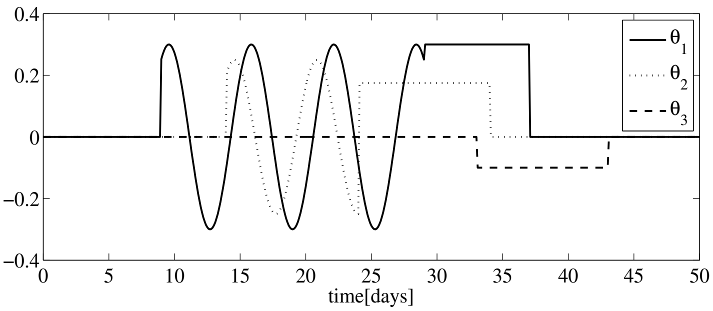

Figure 5: Time varying model uncertainties

system is presented in figure 9. One can see that although a noise is added on the output measurements, output and state estimation are of good quality. One can see that the state and output estimation are of good quality, even if the observer design is based on a reduced system with some simplifying assumptions (fast dynamic considered as algebraic relation, constant premise variables) and even if a measurement noise is added on the system outputs. The VAF (Variance Accounted For) coefficient between two signals is chosen to asses the state estimation quality. The VAF between the i^{th} component of x and $\hat{x}$ is defined by: $VAF_{x_i} = \left[1 - \frac{var(x_i - \hat{x}_i)}{var(x_i)}\right] 100\%$ (a VAF of 100% corresponds to identical signals). The VAF coefficients computed for the original and estimated the state variables are:

$$VAF_x = [92.71;\ 96.45;\ 95.35;\ 95.53;\ 70.25;\ 86.75;\ 99.13;\ 98.73;\ 82.91;\ 96.02]$$

5 Conclusion

In this chapter, some tools for model complexity reduction and their application for observer design were exposed and illustrated on an environmental process. Firstly, a method to rewrite, with no information loss, a generic nonlinear dynamic model as a multi-model with linear submodels and state dependent premise variables was proposed. Since several multi-model forms can be obtained from the original nonlinear model, some choice criteria were recalled in order to select the most suitable form, according to its use (performance analysis, observer / controller design, fault diagnosis etc). Secondly, slow and fast dynamics of the model are identified and separated and the MM is modified according to this separation. The fast dynamics being taken into account as algebraic relations, then a singular multi-model is obtained. In order to estimate the state of such systems, even when all the inputs are not known (unknown input allow to model not only external disturbances or unmeasured system inputs, but also modeling errors), an observer for singular multi-model with unmeasurable premise variables and affected by unknown inputs is proposed. The observer design, aiming at minimizing the influence of the unknown inputs, noise measurements and modeling uncertainties on the estimation, is formulated as an LMI optimization problem. The observer provides the estimation of the slow and fast state variables. Finally,

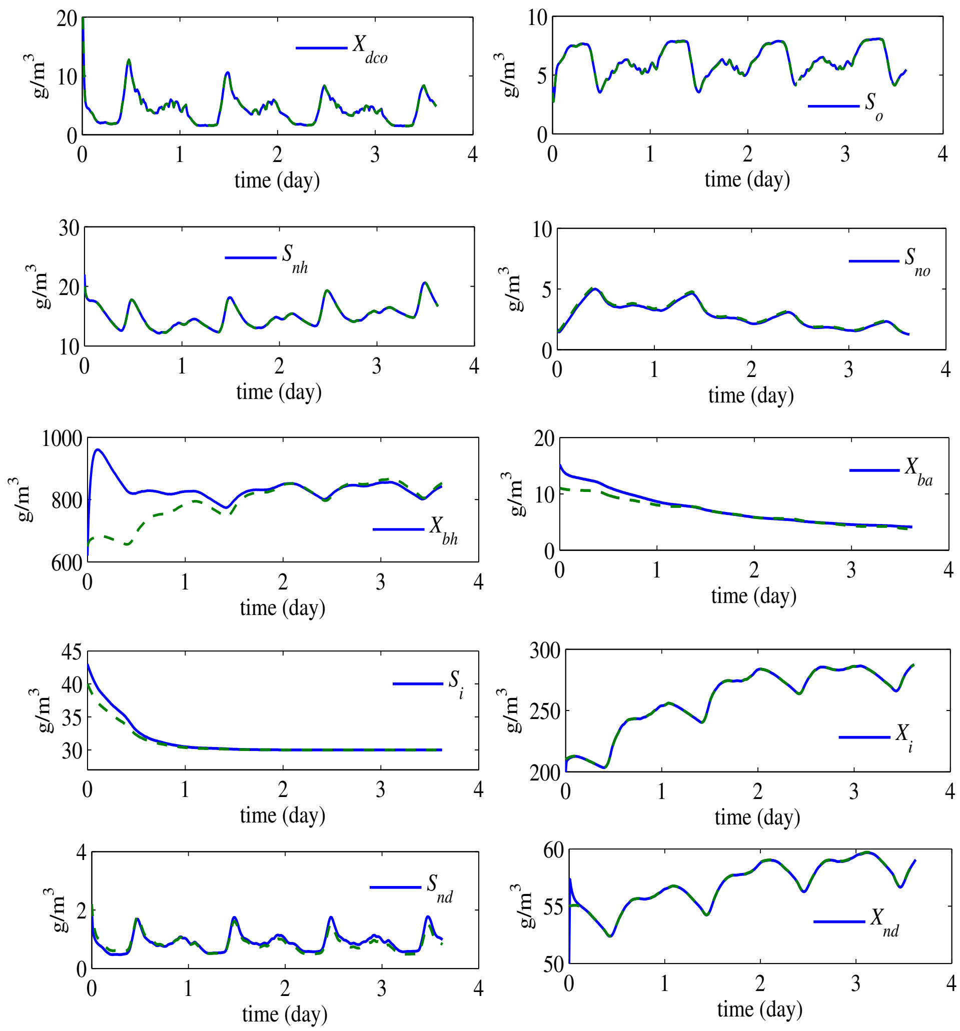

Figure 6: Original and estimated state variables of the ASM1 model

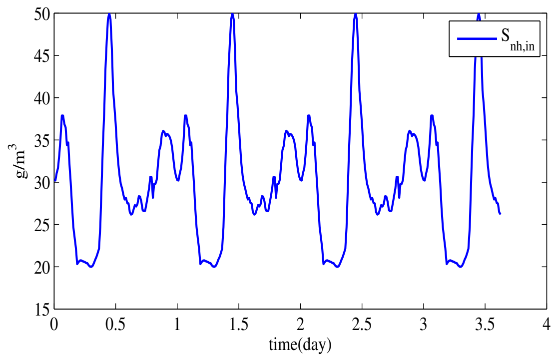

Figure 7: Unknown input of the ASM1 model

an application to a realistic model of a wastewater treatment plant has been exposed and gives good results using the complete ASM1 model.

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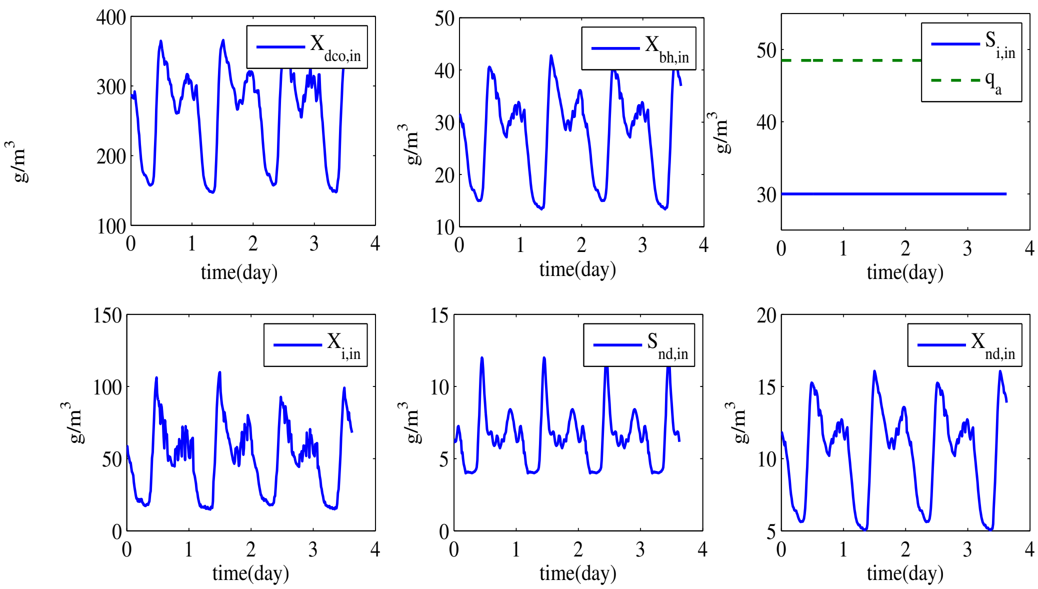

Figure 8: Inputs of the ASM1 model

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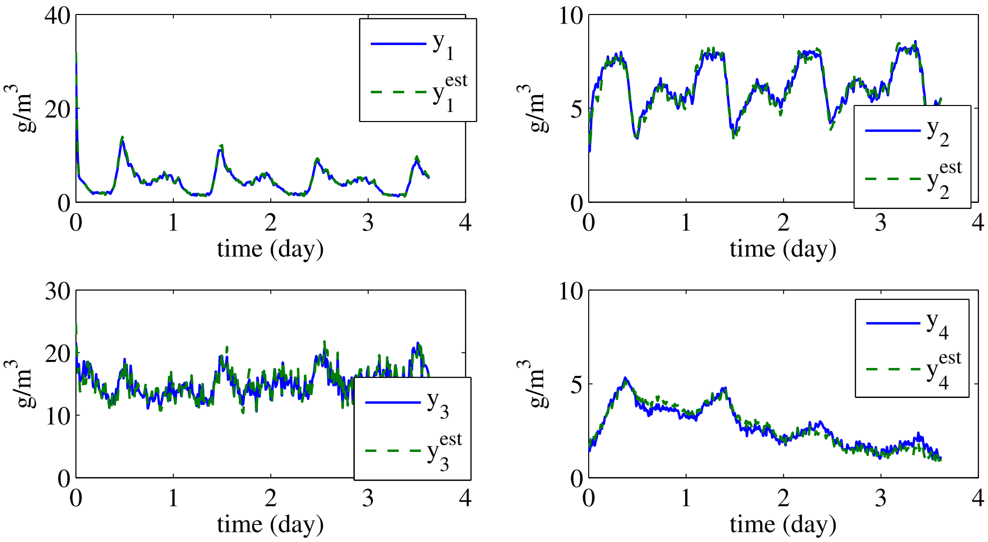

Figure 9: Estimated Outputs for the ASM1 model

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