A descriptor Takagi-Sugeno approach to nonlinear model reduction

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Abstract

In this note, the problem of nonlinear model reduction is addressed. It consists in approximating a given n^{th}-order nonlinear system by a k^{th}-order nonlinear system, where $k < n$. The goal is to compute the reduced order system minimizing the $\mathcal{L}_2$-gain from the input to the difference between the outputs of the original and the reduced systems. For this purpose the nonlinear system generating the approximation error is written under the Takagi-Sugeno formalism and is studied with the use of a multiple Lyapunov function, based on the descriptor approach. The obtained results are expressed in terms of Linear Matrix Inequalities (LMI) and the matrices defining the reduced order system are obtained as a result of LMI problem. Uncertain nonlinear system reduction is also addressed.

Keywords: Model reduction, Takagi-Sugeno systems, $\mathcal{L}_2$ approach, descriptor systems, Linear Matrix Inequality.

1. Introduction

Given an original system (say, of order n), the goal of model order reduction (MOR) is to find a system with a reduced order (say, with $k < n$) that suitably approximates the original system, according to a given norm criterion to be minimized. In general, a reduced order model is sought in order to be analyzed, simulated, or with the objective to design a controller (of reasonable order) for a complex process.

Different methods have arisen in the field of MOR, they can be roughly casted into three families: Krylov subspace method, balance truncated realization and H_∞ approach. Krylov subspace based methods are based on the series expansion at a point of the matrix transfer of the system, see [5]. These

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methods became popular in high dimension circuit simulation where the same structure is repeated (e.g. a transmission line model composed with a large number of identical RLC cells, see [14]) and thus can be projected in a vector base of limited dimension. The main drawback of these methods is that the reduced model approximates well the original transfer mainly around a specified frequency. Krylov MOR are extended to the nonlinear case via linearization around some points [24] or considering a collection of linear models along a state-space trajectory of the original nonlinear system [14, 15], in this last case, the approximation error can be estimated and used to select the linearization points. Another group of MOR methods, appeared after the seminal work of Moore [11] in the early 1980s, is based on Hankel norm approximation and truncated balancing realization (TBR) (see chapters 7 and 8 of [30] or [18] where extended gramians are used) and is suitable for the analysis of LTI systems. Roughly speaking, the minimization of the Hankel norm of the approximation error can be seen as the truncation of the least controllable and observable modes. The advantages of TBR methods are numerous: upper and lower bounds of the approximation error were given in [11] and [8] respectively, stability is preserved, frequency weighting functions can be introduced to improve the precision in some desired frequency range(s) [4, 30], but the high computational cost is a limiting factor in the case of very high dimension systems. The last set of MOR is based on H_∞-optimization and is derived from the H_∞-control theory. Adopting this approach, the reduced order model is seen as a controller designed in order to minimize the H_∞-gain of the transfer from the control input to the approximation error. In the case of LTI systems, Grigoriadis [9] provides necessary and sufficient conditions for the existence of model of k^{th} or zeroth order, both in continuous and discrete time. Since the obtained conditions are not linear (rank constraint), an iterative procedure is proposed. In [3] it is proved that the lower bound of the approximation error provided by H_∞ method is the same one as with TBR and a two-step procedure to compute the reduced order model is proposed. In [7] the reduced order system is obtained in a one-step LMI optimization, but needs a parameter obtained via a balanced realization. Due to the popular LMI formalism, H_∞ model reduction has been extended to several classes of systems: singular systems [27, 29], switched LPV systems [28], switched systems with delay [25].

In this note, an extension of the MOR to nonlinear systems is proposed based on the $\mathcal{L}_2$-approach and on the Takagi-Sugeno (TS) formalism. Since [19], TS systems are extensively investigated due to their approximation properties: indeed, any nonlinear system can be exactly written (i.e. with a zero approximation error) as a TS system on a compact set of $\mathbb{R}^n$. A systematic way to obtain a TS model from a nonlinear model is provided by the sector nonlinearity approach (Chapter 14 of [22], [13]). This property motivates the choice made in this work: considering nonlinear systems under TS form. Beside the approximation property, one of the main advantages of the TS formalism is its closeness to linear formulation. Since a TS system is a time varying blending of LTI submodels, numerous borrowings from the linear theory are possible [22]). Adopting the TS approach, the approximation error system is defined and the approximation error is quantified by its $\mathcal{L}_2$-gain to be minimized by an appropriate choice of the reduced order system. As a result, the computation of the reduced order system is closely related to stability and $\mathcal{L}_2$-norm bound properties. Here are used some recent results on poly-quadratic stability of TS systems [20], improved in [21] to derive LMI conditions for the $\mathcal{L}_2$-norm of a TS system to be bounded by a given positive scalar. Then, a constructive procedure will be detailed in order to compute the reduced order system minimizing the $\mathcal{L}_2$-gain of the error system. The synthesis of k^{th} and zeroth order approximation are treated, as well as MOR of uncertain nonlinear system.

2. Problem formulation and preliminaries

Let a nonlinear system be described by the following TS model

$$\dot{x}(t) = \sum_{i=0}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) \tag{1a}$$

$$y(t) = \sum_{i=1}^{r} h_i(z(t))(C_i x(t) + D_i u(t)) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^q$ are respectively the state vector, the control input, the measured output and the decision variable. Each weighting function $h_i(.)$ quantifies the relative importance of the i^{th} submodel (A_i, B_i, C_i, D_i) in the global nonlinear system (1). These functions satisfy the convex sum properties

$$\sum_{i=1}^{r} h_i(z(t)) = 1 \quad \text{and} \quad 0 \leq h_i(z(t)) \leq 1, \quad \forall t \geq 0, \quad i \in \{1, \ldots, r\} \tag{2}$$

Similarly to [21], the two following assumptions are made in the remaining of this note.

(A1) the decision variable $z(t)$ is real time accessible (e.g. known exogenous signal, input signal).

(A2) the functions h_i satisfy $|\dot{h}_i(z(t))| \leq \Phi_i$, $\forall t > 0$ and $i \in \{1, \ldots, r-1\}$.

The MOR problem can be formulated as finding the reduced order TS system, sharing the same input and decision variable as (1) such that the output of the reduced order system, denoted $y^r(t)$, is as close as possible to $y(t)$, the output of the original system. The reduced order system is proposed to be also in the TS form in order to capture the nonlinear behavior of the original model. The numbers of submodels in both the reduced and the original models are equal to r. In fact, the complexity reduction comes from the choice of a state vector of lower dimension than the one of (1). The proposed reduced system is defined by

$$\dot{x}^r(t) = \sum_{i=0}^{r} h_i(z(t))(A_i^r x^r(t) + B_i^r u(t)) \tag{3a}$$

$$y^r(t) = \sum_{i=1}^{r} h_i(z(t))(C_i^r x^r(t) + D_i^r u(t)) \tag{3b}$$

with $x^r(t) \in \mathbb{R}^k$ with $k < n$ and $y^r(t) \in \mathbb{R}^m$. In order to quantify the distance between the two model outputs, the approximation error system is defined by the difference between the outputs of (1) and (3): $e(t) = y(t) - y^r(t)$. It is readily given by

$$\dot{\bar{x}}(t) = \sum_{i=0}^{r} h_i(z(t))(\bar{A}_i \bar{x}(t) + \bar{B}_i u(t)) \tag{4a}$$

$$e(t) = \sum_{i=1}^{r} h_i(z(t))(\bar{C}_i \bar{x}(t) + \bar{D}_i u(t)) \tag{4b}$$

where $\bar{x}^T(t) = \begin{bmatrix} x^T(t) \; x^{rT}(t) \end{bmatrix}$ and

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & A_i^r \end{bmatrix} \quad \bar{B}_i = \begin{bmatrix} B_i \\ B_i^r \end{bmatrix} \quad \bar{C}_i = \begin{bmatrix} C_i & -C_i^r \end{bmatrix} \quad \bar{D}_i = D_i - D_i^r \tag{5}$$

The approximation error system can be written as a descriptor system

$$\begin{aligned} E^* \dot{x}_a(t) &= \sum_{i=0}^{r} h_i(z(t))(A_i^* x_a(t) + B_i^* u(t)) \\ e(t) &= \sum_{i=1}^{r} h_i(z(t))(C_i^* x_a(t) + D_i^* u(t)) \end{aligned} \tag{6}$$

where $x_a^T(t) = \begin{bmatrix} \bar{x}^T(t) \; \bar{x}^T(t) \end{bmatrix}$ and the matrices A_i^*, B_i^*, C_i^* and D_i^* are defined by $C_i^* = \begin{pmatrix} \bar{C}_i & 0_{m\times(n+k)} \end{pmatrix}$, $D_i^* = \bar{D}_i$ and

$$E^* = \begin{pmatrix} I_{n+k} & 0_{n+k} \\ 0_{n+k} & 0_{n+k} \end{pmatrix} A_i^* = \begin{pmatrix} \bar{A}_i & 0_{n+k} \\ I_{n+k} & -I_{n+k} \end{pmatrix} B_i^* = \begin{pmatrix} \bar{B}_i \\ 0_{(n+k)\times p} \end{pmatrix} \tag{7}$$

As discussed in [21], the interest of the descriptor approach is to introduce some degree of freedom (DOF) in the optimization problem. These DOF are supplementary LMI variables. It must be pointed out, that no impulsive behavior is introduced with this state augmentation since the restriction of $\sum_{i=1}^{r} h_i(z(t))A_i$ in the right kernel of E^* is obviously invertible. The approximation error is given by the $\mathcal{L}_2$-gain of (6) from $u(t)$ to $e(t) = y(t) - y^r(t)$, that is to say the upper bound of the ratio of the energy of these signals (in the LTI case, the $\mathcal{L}_2$-gain coincides with the H_∞-norm of the system). It is well known [2] that a sufficient condition to ensure that the $\mathcal{L}_2$-gain of a system from $u(t)$ to $e(t)$ is less than a given positive scalar γ, is to find a Lyapunov function $V(x_a(t))$ (where $x_a(t)$ is the state vector) verifying

$$\dot{V}(x_a(t)) + e^T(t)e(t) - \gamma^2 u^T(t)u(t) < 0, \quad \forall t > 0 \tag{8}$$

The MOR problem reduces to find the matrices A_i^r, B_i^r, C_i^r and D_i^r (for $i \in \{1, \dots, r\}$) minimizing the $\mathcal{L}_2$-gain of the approximation error system (4) or (6). This minimization is based on the Lyapunov function defined by

$$V(x_a(t)) = x_a^T(t)E^{*T}\left(\sum_{i=1}^{r} h_i(z(t))X_i\right)^{-1} x_a(t) \tag{9}$$

with $X_i^T E^{*T} = E^* X_i \geq 0$ for $i = 1, \dots, r$, proposed in [21] for stability analysis and controller design for TS systems. It will be proved that the matrices X_i are positive definite and thus the inverse in (9) exists.

Notation 1. *M^T stands for the transpose matrix of the matrix M, $M > 0$ (resp. $M < 0$) means that M is a positive (resp. negative) definite matrix. The notation $*$ is used for the blocks induced by symmetry, for any square matrix M, $\mathbb{S}(M)$ is defined by $\mathbb{S}(M) = M + M^T$, I_n is the $n \times n$ identity matrix, 0_n (resp. $O_{n\times m}$) is the $n \times n$ (resp. $n \times m$) null matrix. The matrix $M = diag(M_1, M_2, \dots, M_q)$ is a block diagonal matrix with the blocks M_1, M_2, $\dots$, M_q on its diagonal entries. For any sets of matrices X_i ($i = 1, \dots, r$) and X_{ij} ($i = 1, \dots, r$ and $j = 1, \dots, r$), the polytopic matrices X_h and X_{hh} are respectively defined by*

$$X_h = \sum_{i=1}^{r} h_i(z(t))X_i \qquad X_{hh} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))X_{ij} \tag{10}$$

where the functions $h_i(z(t))$ satisfy the convex sum properties (2).

Before detailing the main results, useful lemmas, taken from [23] and [31] respectively, are recalled.

Lemma 1. *For any $h_i(z(t))$ satisfying* (2) *and any polytopic matrix X_{hh} defined by a double summation according to* (10)*, the inequality $X_{hh} < 0$ holds if the following inequalities are satisfied*

$$X_{ii} < 0, \text{ for } 1 \leq i \leq r \tag{11}$$

$$\frac{1}{r-1}X_{ii} + \frac{1}{2}(X_{ij} + X_{ji}) < 0, \text{ for } 1 \leq i \neq j \leq r \tag{12}$$

Lemma 2. *For any matrices X, $\Sigma(t)$, Y of appropriate dimensions with $I \geq \Sigma^T(t)\Sigma(t)$ and for any positive number τ, it follows*

$$X^T\Sigma^T(t)Y + Y^T\Sigma(t)X \leq \tau X^T X + \tau^{-1}Y^T Y \tag{13}$$

3. Main results

Adopting the $\mathcal{L}_2$-approach, the MOR problem is reduced to an $\mathcal{L}_2$-controller design where the reduced order system is considered as the controller, designed in order to control the approximation error. The order of the controller has to be tunable and less than the one of the original system, as a consequence standard H_∞-control [26] where the controller order is equal to the system order cannot be used, but the descriptor approach is well adapted, since the order of the controller is a degree of freedom.

The proposed result is based on the relaxed stability conditions given in [21] and also on the relaxation introduced by [17] that are modified in order to characterize the $\mathcal{L}_2$-norm bound of a system, and to be able to compute the gains of the reduced order system. In this section, the MOR problem for TS systems is expressed as a problem of minimization under LMI constraints. Once the minimization problem is solved, some of the obtained LMI variables are used to compute the gains of the reduced order system. This method does not provide any *a priori* information on the $\mathcal{L}_2$-gain of the approximation error (no lower or upper bound like in TBR methods for linear models), but the gain is obtained simultaneously with the matrices defining the reduced order model as a result of the optimization process. With this quantitative indicator, it can be appreciated whether the order of the reduced system is sufficiently large to provide a precise approximation of the original system.

3.1. Design of a k^{th} order reduced system

Now the computation of the reduced system of k^{th} order is given in the following theorem.

Theorem 1. *There exists a reduced system (3) of order $k < n$ approximating the system (1) (i.e. minimizing the $\mathcal{L}_2$-gain from $u(t)$ to $e(t)$ in (4)), if there exist matrices $X_i^{11} = X_i^{11T}$, X_i^{31} and $X_i^{33} \in \mathbb{R}^{n\times n}$, matrices X^{12}, X_i^{32} and $X_i^{34} \in \mathbb{R}^{n\times k}$, matrices $X^{22} = X^{22T}$, X_i^{42}, X_i^{44} and $A_{2i}^r \in \mathbb{R}^{k\times k}$, matrices X_i^{41}, X_i^{43} and $A_{1i}^r \in \mathbb{R}^{k\times n}$, matrices $C_{1i}^r \in \mathbb{R}^{m\times n}$ and $C_{2i}^r \in \mathbb{R}^{m\times k}$, matrices $B_i^r \in \mathbb{R}^{k\times p}$ and matrices $D_i^r \in \mathbb{R}^{m\times p}$, minimizing $\bar{\gamma}$ under the LMI constraints (14-17).*

$$\begin{bmatrix} X_i^{11} & X^{12} \\ X^{12T} & X^{22} \end{bmatrix} >0,\ 1 \le i \le r \tag{14}$$

$$X_r^{11} - X_i^{11} \ge 0,\ 1 \le i \le r-1 \tag{15}$$

$$\Theta_{ii} <0,\ 1 \le i \le r \tag{16}$$

$$\frac{1}{r-1}\Theta_{ii} + \frac{1}{2}(\Theta_{ij} + \Theta_{ji}) <0,\ 1 \le i \ne j \le r \tag{17}$$

with

$$\Theta_{ij}=\begin{bmatrix}\Theta_{ij}^{11} & * & * & * & * & * \\ A_{1i}^{r}+(A_iX^{12})^T & \mathbb{S}(A_{2i}^{r}) & * & * & * & * \\ X_j^{11}-X_j^{31} & X^{12}-X_j^{32} & -\mathbb{S}(X_j^{33}) & * & * & * \\ X^{12T}-X_j^{41} & X^{22}-X_j^{42} & -X_j^{43}-X_j^{34T} & -\mathbb{S}(X_j^{44}) & * & * \\ B_i^T & B_i^{rT} & 0 & 0 & -\bar{\gamma}I_p & * \\ C_iX_j^{11}-C_{1i}^{r} & C_iX^{12}-C_{2i}^{r} & 0 & 0 & D_i-D_i^{r} & -I_m\end{bmatrix} \tag{18}$$

where $\Theta_{ij}^{11}=\mathbb{S}(A_iX_j^{11})-\sum_{k=1}^{r-1}\Phi_k(X_k^{11}-X_r^{11})$. *The* $\mathcal{L}_2$*-gain from* $u(t)$ *to* $e(t)$ *is given by* $\gamma=\sqrt{\bar{\gamma}}$ *and the matrices* A_i^r *and* C_i^r *are respectively obtained by*

$$A_i^r=(A_{1i}^rX^{12}+A_{2i}^rX^{22})(X^{12T}X^{12}+X^{22}X^{22})^{-1} \tag{19}$$

$$C_i^r=(C_{1i}^rX^{12}+C_{2i}^rX^{22})(X^{12T}X^{12}+X^{22}X^{22})^{-1} \tag{20}$$

Proof 1. *The following Lyapunov function is considered*

$$V(x_a(t))=x_a^T(t)E^{*T}(X_h)^{-1}x_a(t) \tag{21}$$

where the matrices X_i *are defined by*

$$X_i=\begin{bmatrix}X_i^{11} & X^{12} & 0 & 0 \\ X^{12T} & X^{22} & 0 & 0 \\ X_i^{31} & X_i^{32} & X_i^{33} & X_i^{34} \\ X_i^{41} & X_i^{42} & X_i^{43} & X_i^{44}\end{bmatrix} \tag{22}$$

It can be shown that the LMI constraints (14) *and* (16) *ensure the invertibility of* X_h *(see remark 1, below the proof).*

The time derivative of the Lyapunov function (21) is given by

$$\begin{aligned}\dot{V}(x_a(t))=&\,x_a^T(t)E^{*T}\frac{d}{dt}\left(X_h\right)^{-1}x_a(t)+\dot{x}_a^T(t)E^{*T}\left(X_h\right)^{-1}x_a(t)\\ &+x_a^T(t)E^{*T}\left(X_h\right)^{-1}\dot{x}_a(t)\end{aligned} \tag{23}$$

Due to the definitions of E^* *and* X_h *given by* (7) *and* (22) *respectively, it follows that* $(E^*)^T(X_h)^{-1}=\left((E^*)^T(X_h)^{-1}\right)^T$ *and then the time derivative of* $V(x_a(t))$ *becomes*

$$\dot{V}(x_a(t))=x_a^T(t)E^{*T}\frac{d}{dt}\left(X_h\right)^{-1}x_a(t)+\mathbb{S}(\dot{x}_a^T(t)E^{*T}\left(X_h\right)^{-1}x_a(t)) \tag{24}$$

The definitions of X_i *and* E^* *given in (22) and (7), imply* $E^*X_i=X_i^TE^{*T}$ *and, keeping in mind the positivity of the activating functions, (14) ensures*

$$E^{*T}\left(X_h\right)^{-1}=\left(X_h\right)^{-T}E^*\geq 0 \tag{25}$$

Moreover, derivating (2) *it follows that* $\sum_{i=1}^{r} \dot{h}_i(z(t)) = 0$*, which results in* $\dot{h}_r(z(t)) = -\sum_{i=1}^{r-1} \dot{h}_i(z(t))$*, the second term in (24) can be developed as*

$$\begin{aligned}\frac{d}{dt}\left(\sum_{i=1}^{r} h_i(z(t))X_i\right)^{-1} &= -\left(X_h\right)^{-1}\left(\sum_{i=1}^{r}\frac{d}{dt}h_i(z(t))X_i\right)\left(X_h\right)^{-1} \\ &= -\left(X_h\right)^{-1}\left(\sum_{i=1}^{r-1}\dot{h}_i(z(t))(X_i - X_r)\right)\left(X_h\right)^{-1} \qquad (26)\end{aligned}$$

With (6), (25) and (26), the time derivative of the Lyapunov function becomes

$$\begin{aligned}\dot{V}(x_a(t)) = &-x_a^T(t)\left(X_h\right)^{-T}\left(\sum_{i=1}^{r-1}\dot{h}_i(z(t))E^*(X_i - X_r)\right)\left(X_h\right)^{-1}x_a(t) \\ &+ \mathbb{S}\left(\left(A_h^* x_a(t) + B_h^* u(t)\right)^T\left(X_h\right)^{-1}x_a(t)\right) \qquad (27)\end{aligned}$$

Let $\Gamma(t) = \dot{V}(x_a(t)) + e^T(t)e(t) - \gamma^2 u^T(t)u(t)$ *denote the left hand side of (8) is given by* $\Gamma(t) = \xi^T(t)\mathcal{M}(t)\xi(t)$ *where* $\xi^T(t) = \begin{bmatrix} x_a^T(t) & u^T(t)\end{bmatrix}$ *and* $\mathcal{M}(t)$ *is defined by*

$$\mathcal{M}(t) = \begin{bmatrix} \mathcal{M}_{1a}(t) & * \\ (B_h^*)^T\left(X_h\right)^{-1} & -\gamma^2 I_p\end{bmatrix} + \begin{bmatrix}(C_h^*)^T \\ (D_h^*)^T\end{bmatrix}\begin{bmatrix}(C_h^*)^T \\ (D_h^*)^T\end{bmatrix}^T \qquad (28)$$

with $\mathcal{M}_{1a}(t) = -\left(\left(X_h\right)^{-T}\left(\sum_{k=1}^{r-1}\dot{h}_k(z(t))E^*(X_k - X_r)\right)X_h^{-1} + \mathbb{S}\left(\left(X_h\right)^{-T}A_h^*\right)\right.$*.*
The inequality (8) is satisfied if and only if $\mathcal{M}(t) < 0$*. Applying a Schur complement,* $\mathcal{M}(t) < 0$ *is equivalent to*

$$\begin{bmatrix} \mathcal{M}_{1a}(t) & * & * \\ (B_h^*)^T\left(X_h\right)^{-1} & -\gamma^2 I_p & * \\ C_h^* & D_h^* & -I_m\end{bmatrix} < 0 \qquad (29)$$

Pre- and post-multiplying the inequality (29) by $T = diag((X_h)^T, I_p, I_{n+k})$ *and* T^T *respectively, the following equivalent inequality is obtained*

$$\begin{bmatrix} \mathcal{M}_{1b}(t) & * & * \\ (B_h^*)^T & -\gamma^2 I_p & * \\ C_h^* X_h & D_h^* & -I_m\end{bmatrix} < 0 \qquad (30)$$

with

$$\mathcal{M}_{1b}(t) = -\left(\sum_{k=1}^{r-1}\dot{h}_k(z(t))E^*(X_k - X_r)\right) + \mathbb{S}\left(A_h^* X_h\right) \qquad (31)$$

The inequality (14) is equivalent to $E^*(X_r - X_k) \geq 0$*. With* $|\dot{h}_k(z(t))| \leq \Phi_k$*, it follows*

$$-\sum_{k=1}^{r-1}\dot{h}_k(z(t))E^*(X_k - X_r) \leq -\sum_{k=1}^{r-1}\Phi_k E^*(X_k - X_r) \qquad (32)$$

From (30) *and* (32), *it follows that the $\mathcal{L}_2$-gain of the approximation error is bounded by γ (i.e. (8) is satisfied) if the following inequality holds*

$$\mathcal{M}_{hh} < 0 \tag{33}$$

with

$$\mathcal{M}_{ij} = \begin{bmatrix} -\sum_{k=1}^{r-1} \Phi_k E^*(X_k - X_r) + \mathbb{S}\left(A_i^* X_j\right) & * & * \\ B_i^{*T} & -\gamma^2 I_p & * \\ C_i^* X_j & D_i^* & -I_m \end{bmatrix} \tag{34}$$

Using lemma 1, sufficient conditions are given by

$$\mathcal{M}_{ii} <0,\ 1 \leq i \leq r \tag{35a}$$

$$\frac{1}{r-1}\mathcal{M}_{ii} + \frac{1}{2}(\mathcal{M}_{ij} + \mathcal{M}_{ji}) <0,\ 1 \leq i \neq j \leq r \tag{35b}$$

These inequalities are not linear in the matrices to be determined X_i, A_i^r, B_i^r and C_i^r since some products appear in the previous inequalities. In order to obtain LMI, the following variable changes are defined by $\bar{\gamma} = \gamma^2$ and

$$\begin{bmatrix} A_{1i}^r & A_{2i}^r \end{bmatrix} = A_i^r \begin{bmatrix} X^{12T} & X^{22} \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} C_{1i}^r & C_{2i}^r \end{bmatrix} = C_i^r \begin{bmatrix} X^{12T} & X^{22} \end{bmatrix} \tag{37}$$

With the definition of X_i given by (22) and the variable changes defined by (36-37) it follows

$$A_i^* X_j = \begin{bmatrix} A_i X_j^{11} & A_i X^{12} & 0 & 0 \\ A_{1i}^r & A_{2i}^r & 0 & 0 \\ X_j^{11} - X_j^{31} & X^{12} - X_j^{32} & -X_j^{33} & -X_j^{34} \\ X^{12T} - X_j^{41} & X^{22} - X_j^{42} & -X_j^{43} & -X_j^{44} \end{bmatrix} \tag{38}$$

$$C_i^* X_j = \begin{bmatrix} C_i X_j^{11} - C_{1i}^r & C_i X^{12} - C_{2i}^r & 0 & 0 \end{bmatrix} \tag{39}$$

Thus (34), *with the substitutions* (38) *and* (39) *shows that the inequality (35) becomes (16). The only remaining point is to obtain the matrices of the reduced system. Since the matrices X_i are non-singular (see remark 1 below), the matrix $\begin{bmatrix} X^{12T} & X^{22} \end{bmatrix}$ is full row rank and thus A_i^r and C_i^r can be obtained by right pseudo inversion of $\begin{bmatrix} X^{12T} & X^{22} \end{bmatrix}$ in the variable changes (36) and (37), namely they are respectively given by (19) and (20), which completes the proof.*

Remark 1. *The submatrices obtained by selecting the two first rows and columns of X_i are positive definite, according to* (14). *The submatrices obtained by selecting the two last rows and columns of X_i are also positive definite, according to the LMI* (16) *pre- and post-multiplied by $[0_{n+k}\ I_{n+k}\ 0_{p+m}]$ and its transposed. Consequently, the eigenvalue of the matrices X_i are strictly positive and the matrices X_i are nonsingular.*

3.2. Design of a k^{th} order reduced system for uncertain systems

The result of the previous section can be extended to the case of nonlinear systems affected by bounded time varying model uncertainties. In this case, the system (1) becomes

$$\dot{x}(t) = (A_h + \Delta A(t))x(t) + (B_h + \Delta B(t))u(t) \tag{40a}$$
$$y(t) = (C_h + \Delta C(t))x(t) + (D_h + \Delta D(t))u(t) \tag{40b}$$

where the model uncertainties are defined by

$$\Delta X(t) = M_X \Sigma_X(t) N_X \quad \text{and} \quad 0 \le \Sigma_X^T(t)\Sigma_X(t) \le I \quad \text{for} \quad X \in \{A, B, C, D\} \tag{41}$$

In this case the error system between (40) and (3) is defined by

$$\dot{\bar{x}}(t) = \begin{bmatrix} A_h + \Delta A(t) & 0 \\ 0 & A_h^r \end{bmatrix} \bar{x}(t) + \begin{bmatrix} B_h + \Delta B(t) \\ B_h^r \end{bmatrix} u(t) \tag{42a}$$
$$e(t) = \begin{bmatrix} C_h + \Delta C(t) & -C_h^r \end{bmatrix} \bar{x}(t) + (D_h + \Delta D(t) - D_h^r)u(t) \tag{42b}$$

Despite the presence of model uncertainties, an optimal reduced system (of order $k < n$) can be found, by slightly adapting the result of theorem 1.

Theorem 2. *There exists a reduced system (3) of order $k < n$ approximating the system (40) (i.e. minimizing the $\mathcal{L}_2$-gain from $u(t)$ to $y(t) - y^r(t)$), if there exist matrices $X_i^{11} = X_i^{11T}$, X_i^{31} and $X_i^{33} \in \mathbb{R}^{n\times n}$, matrices X^{12}, X_i^{32} and $X_i^{34} \in \mathbb{R}^{n\times k}$, matrices $X^{22} = X^{22T}$, X_i^{42}, X_i^{44} and $A_{2i}^r \in \mathbb{R}^{k\times k}$, matrices X_i^{41}, X_i^{43} and $A_{1i}^r \in \mathbb{R}^{k\times n}$, matrices $C_{1i}^r \in \mathbb{R}^{m\times n}$ and $C_{2i}^r \in \mathbb{R}^{m\times k}$, matrices $B_i^r \in \mathbb{R}^{k\times p}$, matrices $D_i^r \in \mathbb{R}^{m\times p}$ and positive real numbers τ_{ij}^1, τ_{ij}^2, τ_{ij}^3 and τ_{ij}^4, minimizing $\bar{\gamma}$ under the LMI constraints (14-17), where Θ_{ij} is defined by*

$$\Theta_{ij} = \begin{bmatrix} \tilde{\Theta}_{ij}^{11} & * & * \\ \tilde{\Theta}_{ij}^{21} & \tilde{\Theta}_{ij}^{22} & * \\ \tilde{\Theta}_{ij}^{31} & \tilde{\Theta}_{ij}^{32} & -\Theta_{ij}^{22} \end{bmatrix} \tag{43}$$

with $\Theta_{ij}^{22} = diag(\tau_{ij}^1 I, \tau_{ij}^2 I, \tau_{ij}^3 I, \tau_{ij}^4 I)$ and

$$\begin{aligned}
\tilde{\Theta}_{ij}^{11} &= \begin{bmatrix} \mathbb{S}(A_i X_j^{11}) - \sum_{k=1}^{r-1} \Phi_k (X_k^{11} - X_r^{11}) & * \\ A_{1i}^r + (A_i X^{12})^T & \mathbb{S}(A_{2i}^r) \end{bmatrix} \\
&\quad + diag(\tau_{ij}^1 M_A M_A^T + \tau_{ij}^2 M_B M_B^T, 0) \\
\tilde{\Theta}_{ij}^{21} &= \begin{bmatrix} X_j^{11} - X_j^{31} & X^{12} - X_j^{32} \\ X^{12T} - X_j^{41} & X^{22} - X_j^{42} \\ B_i^T & B_i^{rT} \\ C_i X_j^{11} - C_{1i}^r & C_i X^{12} - C_{2i}^r \end{bmatrix} \\
\tilde{\Theta}_{ij}^{22} &= \begin{bmatrix} -\mathbb{S}(X_j^{33}) & * & * & * \\ -X_j^{43} - X_j^{34T} & -\mathbb{S}(X_j^{44}) & * & * \\ 0 & 0 & -\bar{\gamma} I_p & * \\ 0 & 0 & D_i - D_i^r & -I_m \end{bmatrix} \\
&\quad + diag(0, 0, 0, \tau_{ij}^3 M_C^T M_C + \tau_{ij}^4 M_D^T M_D)
\end{aligned}$$

$$\tilde{\Theta}_{ij}^{31} = \begin{bmatrix} N_A X_j^{31} & N_A X_j^{32} \\ 0 & 0 \\ N_C X_j^{11} & N_C X_j^{12} \\ 0 & 0 \end{bmatrix}$$

$$\tilde{\Theta}_{ij}^{32} = \begin{bmatrix} N_A X_j^{33} & N_A X_j^{34} & 0 & 0 \\ 0 & 0 & N_B & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & N_D & 0 \end{bmatrix}$$

The matrices A_i^r and C_i^r are respectively defined by (19) *and* (20) *and the $\mathcal{L}_2$-gain from $u(t)$ to $e(t) = y(t) - y^r(t)$ is given by $\gamma = \sqrt{\bar{\gamma}}$.*

Proof 2. *Following the proof of theorem 1, the reduced order system is obtained by minimizing γ under the constraint* (33)*. Let us define Θ_{ij}^{11} and Θ_{ij}^{21} by*

$$\Theta_{ij}^{11} = \begin{bmatrix} \tilde{\Theta}_{ij}^{11} & * \\ \tilde{\Theta}_{ij}^{21} & \tilde{\Theta}_{ij}^{22} \end{bmatrix} \tag{44}$$

$$\Theta_{ij}^{21} = \begin{bmatrix} \tilde{\Theta}_{ij}^{31} & \tilde{\Theta}_{ij}^{32} \end{bmatrix} \tag{45}$$

With the model uncertainties (41)*, the condition* (33) *becomes $\mathcal{M}_{hh}^{\Delta} < 0$, with*

$$\mathcal{M}_{ij}^{\Delta} = \mathcal{M}_{ij} + \mathbb{S}(X^T \Sigma^T(t) \Theta_{ij}^{21}) \tag{46}$$

where $\Sigma(t) = diag(\Sigma_A(t), \Sigma_B(t), \Sigma_C(t), \Sigma_D(t))$, $\mathcal{M}_{ij}$ and Θ_{ij}^{21} are defined by (34) *and* (45) *respectively and where X is defined by*

$$X = \begin{bmatrix} M_A^T & 0 & 0 & 0 & 0 & 0 \\ M_B^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_C^T \\ 0 & 0 & 0 & 0 & 0 & M_D^T \end{bmatrix} \tag{47}$$

Using lemma 2, the sufficient conditions (35) *become*

$$\mathcal{M}_{ii} + X^T \Theta_{ii}^{22} X + \Theta_{ii}^{21T} (\Theta_{ii}^{22})^{-1} \Theta_{ii}^{21} < 0, \ 1 \leq i \leq r \tag{48a}$$

$$\begin{aligned} &\frac{1}{r-1} \left(\mathcal{M}_{ii} + X^T \Theta_{ii}^{22} X + \Theta_{ii}^{21T} (\Theta_{ii}^{22})^{-1} \Theta_{ii}^{21} \right) \\ &\quad + \frac{1}{2} \left(\mathcal{M}_{ij} + X^T \Theta_{ij}^{22} X + \Theta_{ij}^{21T} (\Theta_{ij}^{22})^{-1} \Theta_{ij}^{21} \right. \\ &\qquad \left. + \mathcal{M}_{ji} + X^T \Theta_{ji}^{22} X + \Theta_{ji}^{21T} (\Theta_{ji}^{22})^{-1} \Theta_{ji}^{21} \right) < 0, \ 1 \leq i \neq j \leq r \end{aligned} \tag{48b}$$

From (44)*, one can note that $\Theta_{ij}^{11} = \mathcal{M}_{ij} + X^T \Theta_{ij}^{22} X$ and some Schur complements on the terms $(\Theta_{ii}^{22})^{-1}$, $(\Theta_{ij}^{22})^{-1}$ and $(\Theta_{ji}^{22})^{-1}$ in* (48) *allow to obtain the conditions (16-17), with Θ_{ij} defined by* (43)*.*

3.3. Design of a zeroth order reduced system

The simplest reduced system is the zeroth order approximation consisting in a simple gain from the input $u(t)$ to the output $y^r(t)$. In this case, the reduced system is a polytopic matrix D_h^r and the approximation error is given by

$$\dot{x}(t) = A_h x(t) + B_h u(t) \tag{49a}$$
$$e(t) = C_h x(t) + (D_h - D_h^r)u(t) \tag{49b}$$

The computation of the matrices D_i^r is given in the following theorem.

Theorem 3. *There exists a zeroth order system optimally approximating the system (1) (i.e. minimizing the $\mathcal{L}_2$-gain from $u(t)$ to $e(t)$ in (49)), if there exist matrices $X_i^1 = X_i^{1T}$, X_i^2 and $X_i^3 \in \mathbb{R}^{n\times n}$ and matrices $D_i^r \in \mathbb{R}^{m\times p}$, minimizing $\bar{\gamma}$ under the LMI constraints (50-53)*

$$X_i^1 > 0,\ 1 \leq i \leq r \tag{50}$$
$$X_r^1 - X_i^1 \geq 0,\ 1 \leq i \leq r-1 \tag{51}$$
$$\Theta_{ii} < 0,\ 1 \leq i \leq r \tag{52}$$
$$\frac{1}{r-1}\Theta_{ii} + \frac{1}{2}\left(\Theta_{ij} + \Theta_{ji}\right) < 0,\ 1 \leq i \neq j \leq r \tag{53}$$

where Θ_{ij} is defined by

$$\Theta_{ij} = \begin{bmatrix} -\sum_{k=1}^{r-1}\Phi_k(X_k^1 - X_r^1) + \mathbb{S}(A_i X_j^1) & * & * & * \\ X_j^1 - X_j^2 & -\mathbb{S}(X_j^3) & * & * \\ B_i^T & 0 & -\bar{\gamma} I_p & * \\ C_i X_j^1 & 0 & D_i - D_i^r & -I_m \end{bmatrix} \tag{54}$$

Proof 3. *The proof is similar to the one of theorem 1, thus only a sketch is given here. The system* (49) *can be written as the descriptor system* (6) *with $x_a^T(t) = [x^T(t)\ x^T(t)]$, $D_i^* = D_i - D_i^r$ and*

$$E^* = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix} \quad A_i^* = \begin{pmatrix} A_i & 0_n \\ I_n & -I_n \end{pmatrix} \quad B_i^* = \begin{pmatrix} B_i \\ 0 \end{pmatrix} \quad C_i^* = \begin{pmatrix} C_i & 0 \end{pmatrix} \tag{55}$$

The Lyapunov function $V(x_a(t))$ is defined by (21) *with*

$$X_i = \begin{pmatrix} X_i^1 & 0 \\ X_i^2 & X_i^3 \end{pmatrix} \tag{56}$$

From (50)*, the function $V(x_a(t))$ is positive definite. As seen previously, the $\mathcal{L}_2$-gain of the approximation error is bounded by γ if* (33) *holds. Using the system matrices given by* (55) *and the Lyapunov function defined by* (21) *and* (56)*, $\mathcal{M}_{ij}$ in* (34) *becomes Θ_{ij} defined by* (54)*. Then, the inequality* (33) *becomes $\Theta_{hh} < 0$ with Θ_{ij} defined by* (54) *and the LMI conditions (15-17) of theorem 1 become (51-53), which achieves the proof.*

3.4. Relaxed LMI conditions for MOR

In the three previous cases (reduced model of order k for certain or uncertain system and reduced model of zeroth order), the reduced order model is obtained by the minimization of the $\mathcal{L}_2$-gain of the approximation error system. Roughly speaking, this minimization can be summarized by the inequality $\Theta_{hh} < 0$. The most obvious sufficient condition is to impose $\Theta_{ij} < 0$ for $1 \leq i, j \leq r$. This is very conservative and many works were dedicated to the relaxation of such conditions. The relaxation proposed in [23], and recalled in lemma 1, combined with the fuzzy Lyapunov function and the descriptor approach proposed by [21], allow to obtain the LMI (16-17) (resp. (52-53)) of theorems 1 and 2 (resp. theorem 3). Another interesting relaxation scheme is proposed in [17]. This relaxation is based on the convex sum property of the weighting functions and on an appropriate factorization of the multiple sums obtained when multiplying Θ_{hh} by $(\sum_{i=1}^{r} h_i(z(t)))^q = 1$. Using this relaxation, theorem 1 can be reformulated as follows.

Theorem 4. *There exists a reduced system (3) of order $k < n$ approximating the system (1) (i.e. minimizing the $\mathcal{L}_2$-gain from $u(t)$ to $e(t)$ in (4)), if there exist matrices $X_i^{11} = X_i^{11T}$, X_i^{31} and $X_i^{33} \in \mathbb{R}^{n\times n}$, matrices X^{12}, X_i^{32} and $X_i^{34} \in \mathbb{R}^{n\times k}$, matrices $X^{22} = X^{22T}$, X_i^{42}, X_i^{44} and $A_{2i}^r \in \mathbb{R}^{k\times k}$, matrices X_i^{41}, X_i^{43} and $A_{1i}^r \in \mathbb{R}^{k\times n}$, matrices $C_{1i}^r \in \mathbb{R}^{m\times n}$ and $C_{2i}^r \in \mathbb{R}^{m\times k}$, matrices $B_i^r \in \mathbb{R}^{k\times p}$ and matrices $D_i^r \in \mathbb{R}^{m\times p}$, minimizing $\bar{\gamma}$ under the LMI constraints (14-15) and (57-63).*

$$\Theta_{ii} < 0,\ 1 \leq i \leq r \tag{57}$$
$$3\Theta_{ii} + \Theta_{ij} + \Theta_{ji} < 0,\ 1 \leq i \neq j \leq r \tag{58}$$
$$\Theta_{jj} + 3(\Theta_{ii} + \Theta_{ij} + \Theta_{ji}) < 0,\ 1 \leq i \neq j \leq r \tag{59}$$
$$\Theta_{ijk}^1 < 0,\ 1 \leq i < j < k \leq r \tag{60}$$
$$\Theta_{ijk}^2 < 0,\ 1 \leq i \leq j \leq k \leq r \tag{61}$$
$$\Theta_{ijk\ell} < 0,\ 1 \leq i < j < k < \ell \leq r \tag{62}$$
$$\Theta_{ijk\ell m} < 0,\ 1 \leq i < j < k < \ell < m \leq r \tag{63}$$

with

$$\Theta_{ijk}^1 = 6\Theta_{ii} + 3(\Theta_{ij} + \Theta_{ik} + \Theta_{ji} + \Theta_{ki}) + \Theta_{jk} + \Theta_{kj} \tag{64}$$
$$\Theta_{ijk}^2 = \Theta_{ik} + \Theta_{jk} + \Theta_{ki} + \Theta_{kj} + \Theta_{ii} + \Theta_{jj} + 2(\Theta_{ij} + \Theta_{ji}) \tag{65}$$
$$\begin{aligned}\Theta_{ijk\ell} =& 2(\Theta_{ii} + \Theta_{ij} + \Theta_{ik} + \Theta_{i\ell} + \Theta_{ji} + \Theta_{ki} + \Theta_{\ell i}) \\ &+ \Theta_{jk} + \Theta_{j\ell} + \Theta_{kj} + \Theta_{k\ell} + \Theta_{\ell j} + \Theta_{\ell k}\end{aligned} \tag{66}$$
$$\begin{aligned}\Theta_{ijk\ell m} =& \Theta_{ii} + \Theta_{jj} + \Theta_{kk} + \Theta_{\ell\ell} + \Theta_{mm} + \Theta_{ij} + \Theta_{ji} + \Theta_{ik} + \Theta_{ki} \\ &+ \Theta_{i\ell} + \Theta_{\ell i} + \Theta_{im} + \Theta_{mi} + \Theta_{jk} + \Theta_{kj} + \Theta_{j\ell} + \Theta_{\ell j} \\ &+ \Theta_{jm} + \Theta_{mj} + \Theta_{k\ell} + \Theta_{\ell k} + \Theta_{km} + \Theta_{mk} + \Theta_{\ell m} + \Theta_{m\ell}\end{aligned} \tag{67}$$

where Θ_{ij} is defined by (18). The matrices A_i^r and C_i^r are respectively obtained by (19) and (20) and the $\mathcal{L}_2$-gain from $u(t)$ to $e(t) = y(t) - y^r(t)$ is given by $\gamma = \sqrt{\bar{\gamma}}$.

Proof 4. *From the proof of theorem 1, it suffices to prove that the LMI conditions (57-63) imply that* $\Theta_{hh} < 0$*. From* (2)*,* $\Theta_{hh} < 0$ *is equivalent to*

$$\left(\sum_{k=1}^{r} h_k(z(t))\right)^3 \left(\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\Theta_{ij}\right) < 0 \tag{68}$$

Gathering the terms sharing the same combinations of weighting functions (68) *becomes*

$$\begin{aligned}
&\sum_{i=1}^{r} h_i^5(z(t))\Theta_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{r} h_i^4(z(t))h_j(z(t))(3\Theta_{ii}+\Theta_{ij}+\Theta_{ji})\\
&\qquad + \sum_{\substack{i,j,k=1\\ i<j<k}}^{r} h_i^3(z(t))h_j(z(t))h_k(z(t))\Theta_{ijk}^1\\
&\qquad + \sum_{\substack{i,j=1\\ i\neq j}}^{r} h_i^3(z(t))h_j^2(z(t))(\Theta_{jj}+3(\Theta_{ii}+\Theta_{ij}+\Theta_{ji}))\\
&\qquad + \sum_{\substack{i,j,k,\ell=1\\ i<j<k<\ell}}^{r} h_i^2(z(t))h_j(z(t))h_k(z(t))h_\ell(z(t))\Theta_{ijk\ell}\\
&\qquad + \sum_{\substack{i,j,k=1\\ i<j<k}}^{r} h_i^2(z(t))h_j^2(z(t))h_k(z(t))(3\Theta_{ijk}^2)\\
&+ \sum_{\substack{i,j,k,\ell,m=1\\ i<j<k<\ell<m}}^{r} h_i(z(t))h_j(z(t))h_k(z(t))h_\ell(z(t))h_m(z(t))\Theta_{ijk\ell m} < 0
\end{aligned} \tag{69}$$

Since the weighting functions $h_i(z(t))$ *are nonnegative, if each term in the seven sums is negative definite, then* (69) *holds. These terms correspond to the LMI conditions (57-63) which achieves the proof.*

Following the same idea, the reduced system of k^{th} order for uncertain systems, and the zeroth order reduced system can be computed from corollary 1 and 2 respectively. The proofs of these three corollaries are easily deduced from the one of theorem 4 and thus are omitted.

Corollary 1. *There exists a reduced system (3) of order* $k < n$ *optimally approximating the uncertain system (40) (i.e. minimizing the the* $\mathcal{L}_2$*-gain from* $u(t)$ *to* $e(t)$ *in* (42)*), if there exist matrices* $X_i^{11} = X_i^{11T}$*,* X_i^{31} *and* $X_i^{33} \in \mathbb{R}^{n\times n}$*, matrices* X^{12}*,* X_i^{32} *and* $X_i^{34} \in \mathbb{R}^{n\times k}$*, matrices* $X^{22} = X^{22T}$*,* X_i^{42}*,* X_i^{44} *and* $A_{2i}^r \in \mathbb{R}^{k\times k}$*, matrices* X_i^{41}*,* X_i^{43} *and* $A_{1i}^r \in \mathbb{R}^{k\times n}$*, matrices* $C_{1i}^r \in \mathbb{R}^{m\times n}$ *and* $C_{2i}^r \in \mathbb{R}^{m\times k}$*, matrices* $B_i^r \in \mathbb{R}^{k\times p}$*, matrices* $D_i^r \in \mathbb{R}^{m\times p}$ *and positive real numbers* τ_{ij}^1*,* τ_{ij}^2*,* τ_{ij}^3 *and* τ_{ij}^4*, minimizing* $\bar{\gamma}$ *under the LMI constraints (14-15), and (57-63) where* Θ_{ij} *is defined by* (43)*. The matrices* A_i^r *and* C_i^r *are respectively defined by* (19) *and* (20) *and the* $\mathcal{L}_2$*-gain from* $u(t)$ *to* $e(t) = y(t) - y^r(t)$ *is given by* $\gamma = \sqrt{\bar{\gamma}}$*.*

Corollary 2. *There exists a zeroth order system optimally approximating the system* (1) *(i.e. minimizing the the $\mathcal{L}_2$-gain from $u(t)$ to $e(t)$ in* (49)*), if there exist matrices $X_i^1 = X_i^{1T}$, X_i^2 and $X_i^3 \in \mathbb{R}^{n\times n}$ and matrices $D_i^r \in \mathbb{R}^{m\times p}$, minimizing $\bar{\gamma}$ under the LMI constraints (50-51) and (57-63) with Θ_{ij} defined by* (54)*.*

Remark 2. *According to [1, 2, 6] the complexity of solving LMI problems is polynomial in the number of variables and in the dimension of the matrix inequality. Denoting N_d the number of scalar decision variables and M_r the number of rows of the matrix inequality, the complexity of solving the LMI problem is $\mathcal{O}(N_d^2 M_r)$. For instance, in theorems 1 and 3, these numbers are polynomial in the dimensions of the original and reduced systems, namely they are given by: $N_d = n^2\left(\frac{5r}{2}\right) + n\left(\frac{r}{2} + 5kr + k + mr\right) + k^2\left(3r + \frac{1}{2}\right) + k\left(\frac{1}{2} + mr + pr\right) + 1$ and $M_r = n(2r^2+2r-1)+k(2r^2+r)+r^2(p+m)$, consequently the overall complexity is $\mathcal{O}(n^5, k^5)$. Due to this polynomial complexity and also to the fact that LMI-based methods are not constructive, the proposed results are not relevant for very large scale systems (thousands of state variables) but are rather devoted to the order reduction of nonlinear systems in order to ease simulations or the implementation of a reduced controller/observer. It should be kept in mind that T-S modeling can capture any nonlinear behaviors, and that model/controller/observer order reduction in the general framework of nonlinear systems is not a trivial issue.*

4. Numerical example

The results presented in theorem 1 are applied in order to compute a fifth order nonlinear system approximation of a ninth order nonlinear system with $r = 3$ subsystems, $m = 1$ output and $p = 2$ inputs, defined by (1) with

$$A_1 = \begin{bmatrix} 464 & -256 & -512 & 0 & -512 & 0 & 0 & 0 & 512 \\ 176 & -224 & -176 & -64 & -176 & 0 & 64 & -64 & 176 \\ 152 & -128 & -192 & 0 & -152 & 0 & 0 & -32 & 184 \\ -96 & 64 & 128 & 32 & 96 & 0 & -33 & 33 & -96 \\ 102 & 124 & -108 & 64 & -134 & -16 & -63 & 99 & 110 \\ 378 & -100 & -408 & -64 & -422 & -4 & 65 & -57 & 406 \\ -42 & -36 & 60 & 64 & 42 & 0 & -64 & 76 & -58 \\ -42 & -36 & 60 & 64 & 42 & 0 & -63 & 75 & -58 \\ -130 & 124 & 132 & 64 & 146 & -16 & -63 & 67 & -138 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 696 & -512 & -769 & 0 & -769 & 0 & 0 & 0 & 769 \\ 361 & -433 & -361 & -128 & -361 & 0 & 128 & -128 & 361 \\ 453 & -256 & -594 & 0 & -453 & 0 & 0 & -128 & 581 \\ -161 & 129 & 193 & 80 & 161 & 0 & -81 & 81 & -161 \\ -56 & 203 & 79 & 128 & -1 & -16 & -126 & 212 & -76 \\ 399 & -185 & -406 & -128 & -465 & -7 & 130 & -81 & 401 \\ -112 & -85 & 179 & 128 & 112 & 0 & -128 & 185 & -176 \\ -112 & -85 & 179 & 128 & 112 & 0 & -126 & 184 & -176 \\ -116 & 203 & 71 & 128 & 132 & -16 & -126 & 84 & -81 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 348 & -256 & -385 & 0 & -385 & 0 & 0 & 0 & 385 \\ 180 & -217 & -180 & -64 & -180 & 0 & 64 & -64 & 180 \\ 226 & -128 & -297 & 0 & -226 & 0 & 0 & -64 & 290 \\ -80 & 64 & 96 & 40 & 80 & 0 & -41 & 41 & -80 \\ -28 & 101 & 40 & 64 & 0 & -8 & -63 & 106 & -38 \\ 200 & -93 & -203 & -64 & -233 & -4 & 65 & -41 & 201 \\ -56 & -43 & 90 & 64 & 56 & 0 & -64 & 93 & -88 \\ -56 & -43 & 90 & 64 & 56 & 0 & -63 & 92 & -88 \\ -58 & 101 & 36 & 64 & 66 & -8 & -63 & 42 & -41 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.29 & -0.15 \\ -0.021 & -0.01 \\ -0.035 & -0.017 \\ -0.015 & -0.0074 \\ -0.24 & -0.12 \\ -0.22 & -0.11 \\ -0.039 & -0.02 \\ -0.039 & -0.02 \\ -0.013 & -0.0064 \end{bmatrix} \quad B_2 = \begin{bmatrix} -1.81 & -0.903 \\ -0.182 & -0.0912 \\ -0.256 & -0.128 \\ -0.0704 & -0.0352 \\ -1.38 & -0.689 \\ -1.27 & -0.634 \\ -0.281 & -0.141 \\ -0.281 & -0.141 \\ -0.0647 & -0.0323 \end{bmatrix} \quad B_3 = \begin{bmatrix} -0.24 & -0.12 \\ -0.025 & -0.012 \\ -0.035 & -0.017 \\ -0.0095 & -0.0048 \\ -0.19 & -0.093 \\ -0.17 & -0.086 \\ -0.038 & -0.019 \\ -0.038 & -0.019 \\ -0.0087 & -0.0044 \end{bmatrix}$$

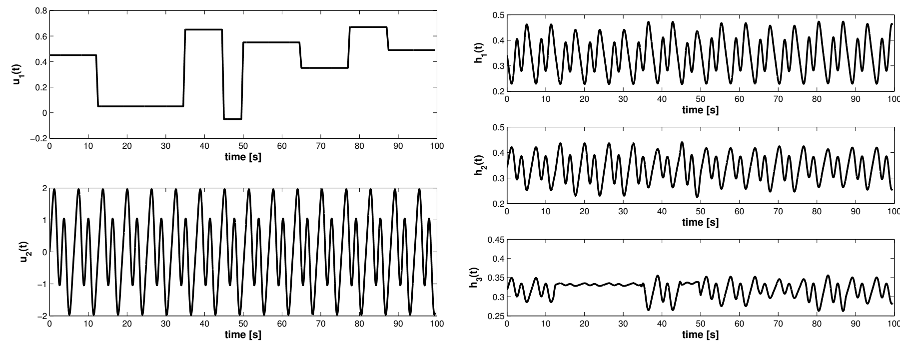
$$C_1 = \begin{bmatrix} -0.25 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & -0.5 \end{bmatrix} \quad D_1 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -0.27 & 0 & 0.54 & 0 & 0.54 & 0 & 0 & 0 & -0.54 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} -1 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & -2 \end{bmatrix} \quad D_3 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}$$

The weighting functions, depending on the input signals, are defined by $w_1(t) = (tanh((u_1(t)u_2(t))/6) + 1)$, $w_2(t) = (tanh((u_1(t) + u_2(t))/6) + 1)$, $w_3(t) = (tanh((u_1(t) - u_2(t))/6) + 1)$ and $h_i(t) = (w_i(t))/(\sum_{k=1}^{r} w_k(t))$. The input signals and the weighting functions are depicted on figures 1(a) and 1(b) respectively. The upper bound on the norm of the time derivative of the weighting functions are $\Phi_k = 0.3$. The LMI problem given by (14-17) is solved with Matlab and the solver Yalmip. The reduced order system is defined by (3) with

$$A_1^r = \begin{bmatrix} -5.54 & 2.7164 & 8.1067 & -0.69063 & 0.077528 \\ 5.7587 & -42.465 & -22.908 & 4.5168 & -2.5344 \\ 46.068 & -81.293 & -172.48 & -6.7698 & -12.661 \\ 41.606 & 15.058 & -19.34 & -145.87 & 8.4247 \\ 163.19 & -372.05 & -479 & -23.154 & -188.39 \end{bmatrix}$$



(a) System inputs $u_1(t)$ and $u_2(t)$ (b) Weighting functions $h_i(u(t))$

Figure 1: System inputs and weighting functions

$$A_2^r = \begin{bmatrix} -13.498 & 11.606 & 3.9158 & 0.68196 & -0.18676 \\ 16.581 & -31.974 & -30.24 & 7.4457 & -1.3846 \\ 23.666 & -63.631 & -148.08 & 47.48 & -16.534 \\ -1.3843 & 76.956 & 224.18 & -126.8 & 30.756 \\ 18.164 & -112.2 & -640.48 & 171.22 & -212.89 \end{bmatrix}$$

$$A_3^r = \begin{bmatrix} -4.693 & 3.059 & 3.588 & -0.8411 & 0.293 \\ 5.705 & -26.76 & -9.257 & 5.695 & -0.7484 \\ 10.86 & -6.097 & -32.66 & 4.459 & -3.99 \\ -4.44 & 36 & 7.816 & -32.68 & 3.524 \\ 37.42 & -31.64 & 24.39 & -36.11 & -30.74 \end{bmatrix}$$

$$B_1^r = \begin{bmatrix} 0.439 & 0.219 \\ 0.215 & 0.108 \\ -0.493 & -0.246 \\ 0.598 & 0.299 \\ 5.39 & 2.7 \end{bmatrix} \quad B_2^r = \begin{bmatrix} 0.765 & 0.382 \\ -0.598 & -0.299 \\ -0.567 & -0.284 \\ -0.371 & -0.185 \\ 3.18 & 1.59 \end{bmatrix} \quad B_3^r = \begin{bmatrix} 0.168 & 0.0838 \\ -0.141 & -0.0703 \\ -0.0724 & -0.0362 \\ 0.169 & 0.0846 \\ 1.88 & 0.94 \end{bmatrix}$$

$$C_1^r = \begin{bmatrix} -0.43 & 0.53 & 0.68 & -0.061 & 0.031 \end{bmatrix} \quad D_1^r = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}$$
$$C_2^r = \begin{bmatrix} -0.38 & 0.34 & 0.16 & -0.031 & -0.0085 \end{bmatrix} \quad D_2^r = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}$$
$$C_3^r = \begin{bmatrix} -0.62 & 0.2 & 0.24 & -0.035 & 0.02 \end{bmatrix} \quad D_3^r = \begin{bmatrix} 0.19 & 0.1 \end{bmatrix}$$

The original and reduced system outputs are depicted on figure 2.

5. Conclusion

In this note, LMI conditions have been proposed to design a reduced order Takagi-Sugeno system that approximates a nonlinear system of higher order,

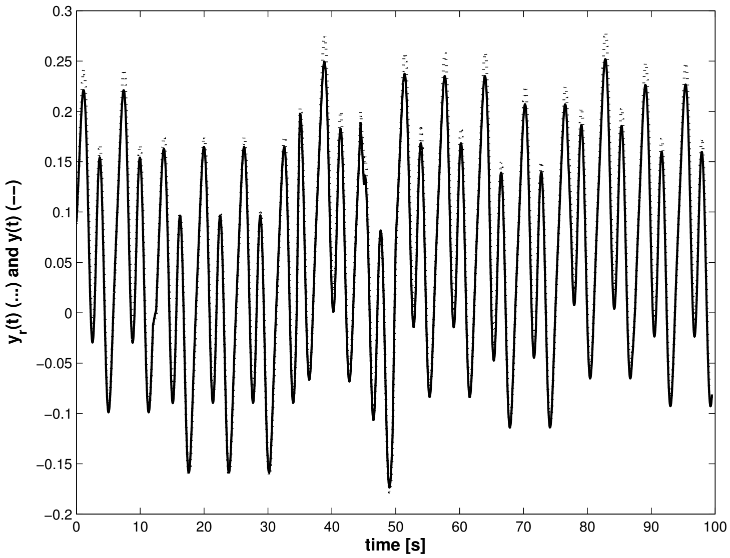


Figure 2: Original and reduced system outputs $y(t)$ (solid line) and $y_r(t)$ (dashed line)

in the continuous time case. The approximation error is quantified by the $\mathcal{L}_2$-gain of the system generating the approximation error. The optimal solution to model order reduction (MOR) is found by minimizing this gain. An extension of the MOR to uncertain nonlinear systems is proposed. The special case of zeroth order reduced system is also treated. Finally, a particular attention is paid to conservatism reduction of the obtained LMI conditions. Future works may concern the introduction of frequency weighting transfer functions in order to highlight a particular operating frequency range in which the original system should be precisely approximated by the reduced one and the extension of the presented result to the discrete time case, using non-quadratic Lyapunov function like in [10], or the study of MOR for T-S systems with activating functions depending on the state variables, using the results of [12, 16].

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