A descriptor Takagi-Sugeno approach to frequency weighted nonlinear model reduction

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Outline of the talk

Problem statement and background

Nonlinear model order reduction

Weighted nonlinear model order reduction

Numerical example

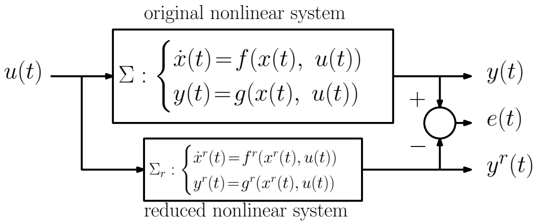
Concluding remarks

Problem statement: model order reduction (MOR)

- Model order reduction of a dynamic nonlinear system

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases} \quad =?\Rightarrow \quad \Sigma^r : \begin{cases} \dot{x}^r(t) = f^r(x^r(t), u(t)) \\ y^r(t) = g^r(x^r(t), u(t)) \end{cases}$$

- reduced order: $dim(x^r) = k < dim(x) = n$
- output approximation: $y^r(t) \simeq y(t)$
- approximation error minimization: $\min_{\Sigma_r} e(t)$



Some background on MOR

- Existing techniques for MOR
  - Krylov subspaces
  - Hankel norm approximation
  - $\mathcal{H}_\infty$-approach

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    series expansion of the matrix transfer of the linear(ized) system
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    truncation of the less controllable and observable states
    $+$ upper bound of the approximation error
    $-$ for linear systems
  - $\mathcal{H}_\infty$-approach

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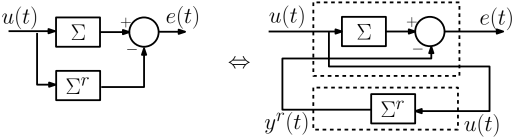

reduced system $\sim$ controller of the approximation error
$+$ upper bound of the approximation error
$+$ possible extension to nonlinear systems: $\mathcal{H}_\infty \to \mathcal{L}_2$

Some background on the Takagi-Sugeno approach

- Any nonlinear system can be written as a Takagi-Sugeno system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases} \Rightarrow \begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{r} h_i(z(t))(C_i x(t) + D_i u(t)) \end{cases}$$

where

- $z(t)$ is the decision variable
- $h_i(z(t))$ are the activating functions
- the activating functions satisfy the convex sum properties:

$$0 \leq h_i(z(t)) \leq 1 \quad \text{and} \quad \sum_{i=1}^{r} h_i(z(t)) = 1$$

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- Assumptions
  - the decision variables $z(t)$ are accessible
  - the derivative of the activating functions are lower bounded:

$$|\dot{h}_i(z(t))| \geq \Phi_i, \quad \forall t > 0, \; i \in \{1, \dots, r-1\}$$

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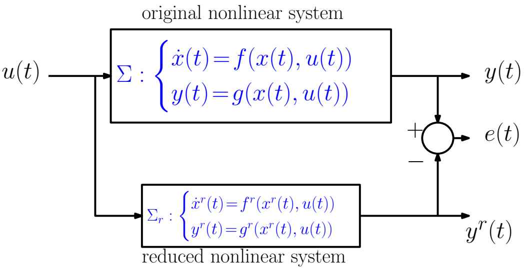
- Notations

$$X_h = \sum_{i=1}^{r} h_i(z(t)) X_i \text{ and } X_{hh} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) X_{ij}$$

Descriptor Takagi-Sugeno approach to nonlinear MOR

- **TS approach of the system nonlinearity**
- $\mathcal{H}_\infty$-approach of the MOR
- Descriptor approach
- Nonquadratic Lyapunov function
- Tuan's relaxation

Both original and reduced nonlinear systems are represented by Takagi-Sugeno models



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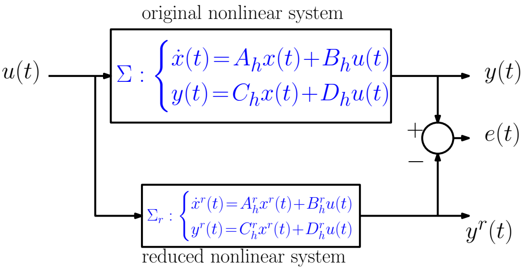

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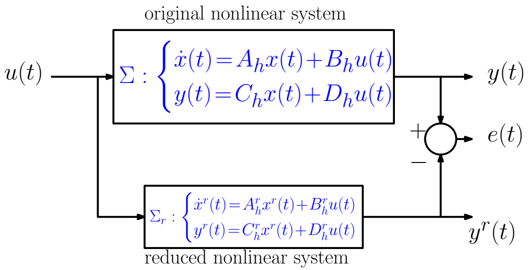
Nonlinear reduced order model Σ^r seen as a controller of e by y_r
$\Rightarrow$ MOR $\sim$ Find $(A_i^r, B_i^r, C_i^r, D_i^r)$ minimizing the $\mathcal{L}_2$-gain from u to e



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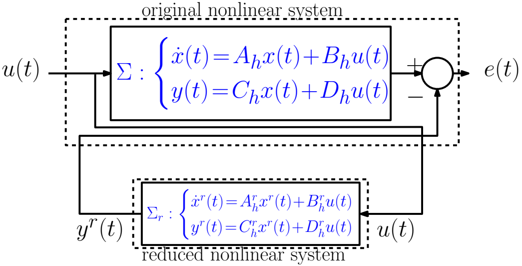

- TS approach of the system nonlinearity
- $\mathcal{H}_\infty$-approach of the MOR
- **Descriptor approach**
- Nonquadratic Lyapunov function
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The closed-loop system from $u(t)$ to $e(t)$:

$$\begin{cases} \dot{\bar{x}} = \bar{A}_h \bar{x} + \bar{B}_h u \\ e = \bar{C}_h \bar{x} + \bar{D}_h u \end{cases}, \bar{x} = \begin{pmatrix} x \\ x^r \end{pmatrix}, \bar{A}_i = \begin{pmatrix} A_i & 0 \\ 0 & A_i^r \end{pmatrix}, \bar{B}_i = \begin{pmatrix} B_i \\ B_i^r \end{pmatrix}, \bar{C}_i = \begin{pmatrix} C_i & -C_i^r \end{pmatrix}, \bar{D}_i = D_i - D_i^r$$

is augmented into an equivalent descriptor TS:

$$\begin{pmatrix} I_{n+k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{x}} \end{pmatrix} = \begin{pmatrix} \bar{A}_h & 0 \\ I_{n+k} & -I_{n+k} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix} + \begin{pmatrix} \bar{B}_h \\ 0 \end{pmatrix} u$$

$$e = \begin{pmatrix} \bar{C}_h & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix} + \bar{D}_h u$$

Conservatism reduction:

- additional slack variables in the $\mathcal{L}_2$-control design
- decoupling Lyapunov and system matrices (including A_i^r, B_i^r, C_i^r and D_i^r)

Descriptor Takagi-Sugeno approach to nonlinear MOR

- TS approach of the system nonlinearity
- $\mathcal{H}_\infty$-approach of the MOR
- Descriptor approach (conservatism reduction)
- **Nonquadratic Lyapunov function**
- Tuan's relaxation

Following *(Tanaka et al., IEEE TFS, 2007)*, define a nonquadratic Lyapunov function:

$$V(t) = \begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix}^T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} X_h^{11} & 0 \\ X_h^{21} & X_h^{22} \end{pmatrix}^{-1} \begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix}$$

Find $(A_i^r, B_i^r, C_i^r, D_i^r)$ and the X_i^{**} minimizing γ, under the constraints:

- positivity of the Lyapunov function: $0 < V(t)$
- $\mathcal{L}_2$-norm bound: $0 > \dot{V}(t) + e^T e - \gamma^2 u^T u$

Conservatism reduction:

- multiple Lypaunov matrices

Descriptor Takagi-Sugeno approach to nonlinear MOR

- TS approach of the system nonlinearity
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- **Tuan's relaxation**

Following *(Tuan et al., IEEE TFS, 2001)*, it is known that:

$$X_{hh} < 0$$

is implied by:

$$\begin{cases} 0 > X_{ii}, & 1 \leq i \leq r \\ 0 > \frac{1}{r-1} X_{ii} + \frac{1}{2}(X_{ij} + X_{ji}), & 1 \leq i \neq j \leq r \end{cases}$$

Main result: computation of Σ^r

There exists Σ^r of order $k < n$ minimizing the $\mathcal{L}_2$-gain from $u(t)$ to $e(t)$, if there exists matrices X_i^{**}, A_{*i}^r, C_{*i}^r, B_i^r and D_i^r, minimizing $\bar{\gamma}$ under the LMI

$$0 < \begin{bmatrix} X_i^{11} & X^{12} \\ X^{12T} & X^{22} \end{bmatrix} \qquad 0 > \Theta_{ii}$$

$$0 \geq X_i^{11} - X_r^{11} \qquad 0 > \frac{1}{r-1}\Theta_{ii} + \frac{1}{2}(\Theta_{ij} + \Theta_{ji})$$

with Θ_{ij} linear in the LMI variables X_i^{**}, A_{1i}^r, A_{2i}^r, C_{1i}^r, C_{2i}^r, B_i^r, D_i^r and $\bar{\gamma}$.

$$\Theta_{ij} = \begin{bmatrix} \mathbb{S}(A_i X_j^{11}) - \sum_{k=1}^{r-1} \Phi_k (X_k^{11} - X_r^{11}) & * & * & * & * & * \\ A_{1i}^r + (A_i X^{12})^T & \mathbb{S}(A_{2i}^r) & * & * & * & * \\ X_j^{11} - X_j^{31} & X^{12} - X_j^{32} & -\mathbb{S}(X_j^{33}) & * & * & * \\ X^{12T} - X_j^{41} & X^{22} - X_j^{42} & -X_j^{43} - X_j^{34T} & -\mathbb{S}(X_j^{44}) & * & * \\ B_i^T & B_i^{rT} & 0 & 0 & -\bar{\gamma} I & * \\ C_i X_j^{11} - C_{1i}^r & C_i X^{12} - C_{2i}^r & 0 & 0 & D_i - D_i^r & -I \end{bmatrix}$$

with: $\mathbb{S}(M) = M + M^T$.

LMI Design of the reduced order system

Main result: computation of Σ^r

There exists Σ^r of order $k < n$ minimizing the $\mathcal{L}_2$-gain from $u(t)$ to $e(t)$, if there exists matrices X_i^{**}, A_{*i}^r, C_{*i}^r, B_i^r and D_i^r, minimizing $\bar{\gamma}$ under the LMI

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$$

with Θ_{ij} linear in the LMI variables X_i^{**}, A_{1i}^r, A_{2i}^r, C_{1i}^r, C_{2i}^r, B_i^r, D_i^r and $\bar{\gamma}$.

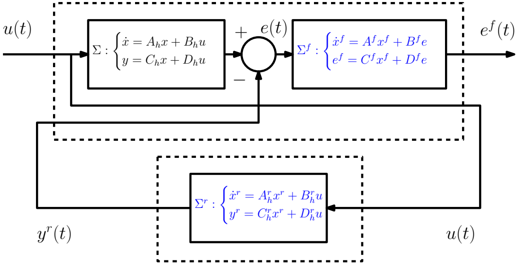
- Reduced system Σ^r: B_i^r and D_i^r are LMI variables and A_i^r and C_i^r are obtained by:

$$
\begin{aligned}
A_i^r &= (A_{1i}^r X^{12} + A_{2i}^r X^{22})(X^{12T} X^{12} + X^{22} X^{22})^{-1} \\
C_i^r &= (C_{1i}^r X^{12} + C_{2i}^r X^{22})(X^{12T} X^{12} + X^{22} X^{22})^{-1}
\end{aligned}
$$

- $\mathcal{L}_2$-gain from $u(t)$ to $e(t)$: $\gamma = \sqrt{\bar{\gamma}}$

Weighted nonlinear model order reduction

- Frequency weighting function Σ^f can be introduced to relax the $\mathcal{L}_2$-constraint in some frequency range(s)



- The same machinery is applied to the augmented system

$$\begin{pmatrix} \dot{x}^f \\ \dot{x} \\ \dot{x}^r \end{pmatrix} = \begin{pmatrix} A^f & B^f C_h & -B^f C_h^r \\ 0 & A_h & 0 \\ 0 & 0 & A_h^r \end{pmatrix} \begin{pmatrix} x^f \\ x \\ x^r \end{pmatrix} + \begin{pmatrix} B^f (D_h - D_h^r) \\ B_h \\ B_h^r \end{pmatrix} u$$

$$e^f = \begin{pmatrix} C^f & D^f C_h & -D^f C_h^r \end{pmatrix} \begin{pmatrix} x^f \\ x \\ x^r \end{pmatrix} + D^f (D_h - D_h^r) u$$

Consider the system of order $n = 5$ with $r = 3$ submodels:

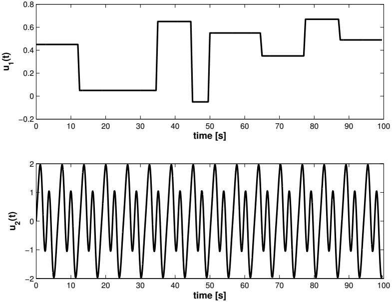
$$A_1 = \begin{bmatrix} -12 & -32 & 38 & -38 & 0 \\ 1.67 & -26 & -2.33 & -1.67 & 0 \\ -1.33 & -24 & -6.33 & 1.33 & 8 \\ -4.33 & -8 & 40.67 & -45.67 & 8 \\ 1.67 & -25 & -2.33 & -1.67 & -1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0.57 & 0.41 \\ -0.11 & -0.037 \\ 0.98 & 0.45 \\ 0.92 & 0.44 \\ 0.28 & 0.08 \end{bmatrix} \quad C_1^T = \begin{bmatrix} 0.667 \\ 0 \\ -1.33 \\ 1.33 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -7.33 & -32 & 24.67 & -24.67 & 0 \\ 6 & -41 & 2 & -6 & 0 \\ 2.33 & -18 & -15.67 & -2.33 & 2 \\ 6 & -2 & 20 & -38 & 2 \\ 6 & -33 & 2 & -6 & -8 \end{bmatrix} \quad B_2 = \begin{bmatrix} -0.5 & -0.26 \\ -0.22 & -0.13 \\ 0.21 & 0.084 \\ 0.25 & 0.11 \\ -0.22 & -0.15 \end{bmatrix} \quad C_2^T = \begin{bmatrix} 0.17 & \\ 0 & \\ -0.33 & \\ 0.33 & 0 \end{bmatrix}$$

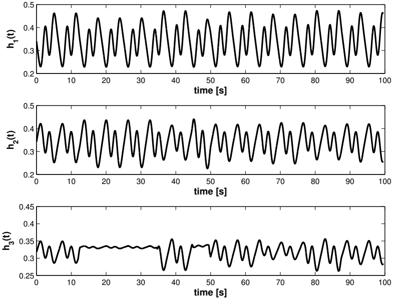
$$A_3 = \begin{bmatrix} -15.33 & -32 & 34.67 & -34.67 & 0 \\ 0.333 & -23 & -7.67 & -0.333 & 0 \\ -3 & -24 & -8 & 3 & 8 \\ -4.33 & -8 & 40.67 & -45.67 & 8 \\ 0.333 & -22 & -7.67 & -0.333 & -1 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0.41 & 0.25 \\ -0.19 & -0.13 \\ 0.62 & 0.42 \\ 0.52 & 0.36 \\ 0.088 & 0.09 \end{bmatrix} \quad C_3 = \begin{bmatrix} 0.33 \\ 0 \\ -0.67 \\ 0.67 \\ 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.005 & 0.005 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0.004 & 0.002 \end{bmatrix} \quad D_3 = \begin{bmatrix} 0.004 & 0.002 \end{bmatrix}$$

System inputs



Weighting functions



$$w_1(t) = (tanh((u_1(t)u_2(t))/6) + 1)$$
$$w_2(t) = (tanh((u_1(t) + u_2(t))/6) + 1)$$
$$w_3(t) = (tanh((u_1(t) - u_2(t))/6) + 1)$$
$$h_i(t) = \frac{w_i(t)}{\sum_{k=1}^{r} w_k(t)}$$

Numerical example

Nonlinear MOR results, for $k = 2$

- without frequency weighting
  $\mathcal{L}_2$-gain from u to e:

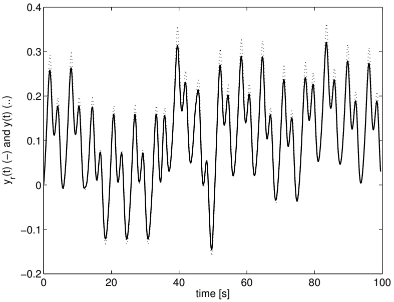
  $$\gamma = 0.14$$

- with frequency weighting:

  $$W^f(s) = \frac{0.0625(s+0.005)^2(s+2000)^2}{(s+0.02)^2(s+500)^2}$$

  $\mathcal{L}_2$-gain from u to e:

  $$\gamma = 0.08$$



Concluding remarks

– No a priori upper/lower bound on the approximation error can be set...
... but γ is a result of the LMI problem

[1] B. Marx, A descriptor Takagi-Sugeno approach to nonlinear model reduction, *Linear Algebra and its Applications*, 479, 52-72,2015

Concluding remarks

- No a priori upper/lower bound on the approximation error can be set...
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- Dedicated to MOR of *not so large* scale systems:
  the overall LMI problem complexity is $\mathcal{O}(N_d^2 M_r) \sim \mathcal{O}(n^5, k^5)$
  with N_d scalar decision variables and M_r rows in the matrix inequality

$$N_d = n^2\left(\frac{5r}{2}\right) + n\left(\frac{r}{2} + 5kr + k + n_y r\right) + k^2\left(3r + \frac{1}{2}\right) + k\left(\frac{1}{2} + n_y r + n_u r\right) + 1$$

$$M_r = n(2r^2 + 2r - 1) + k(2r^2 + r) + r^2(n_u + n_y)$$

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$$M_r = n(2r^2+2r-1)+k(2r^2+r)+r^2(n_u+n_y)$$

+ The reduced system order k, is tunable
+ The special case of a 0^{th} order approximation is easily treated[1]
+ Extension of the results to time varying uncertain systems is easy[1]
+ Polya's scheme of *(Sala and Ariño, Fuzzy Sets Syst., 158(24), 2671-2686,2007)* can be applied to obtain relaxed LMI conditions[1]

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