State and Parameter Estimation for Time-varying Systems: a Takagi-Sugeno Approach

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Abstract: The contribution of this paper is to propose a systematic approach to the observer design for linear time-varying systems. It is based on the exact rewriting of the original time-varying system into a polytopic linear model (PLM). This transformation uses the sector nonlinearity approach based on the convex polytopic transformation. Then a joint state and parameter observer can be designed for the PLM and the estimation errors convergence are proved.

Keywords: Time-varying linear systems, polytopic linear models, sector nonlinearity approach, convex polytopic transformation, state and parameter observer, linear matrix inequality.

1. INTRODUCTION

Due to an increasing demand for higher performances, safety and reliability, model-based fault diagnosis (detection and isolation) is of practical importance and has received considerable interest these past years. The state estimation problem can be viewed as the heart of control systems and model-based diagnosis (Chen and Patton [1999]), (Basseville [1998]). The design of observers, reconstructing state variables out of a limited set of measurements, is a possible approach for dealing with the measurement problem. But, due to time-varying behaviour of dynamical processes, the introduction of time-varying parameters in the system models leads to a higher level of complexity with more challenging problems in estimation. In this case, conventional observers essentially developped for time invariant systems cannot directly be used, and so-called adaptive observers developed for joint state and unknown parameter estimation has to be implemented.

However, the difficulty in estimating such models is augmented by the fact that one has no idea regarding the way that the parameters may vary. In the present work, a focus is made on the linear time varying parameter (LPV) systems where the parameter variations are mostly inaccessible (non measurable) and may be considered as faults (acting as disturbances or/and uncertainties).

Some methods have been published in this subject. They assume the existence of some Lyapunov function satisfying particular conditions and there is no systematic way to check their applicability to a given system (Zhang and Xu [2001]), (Rajamani and Hedrich [1995]), (Besançon [2000]), (Cho and Rajamani [1997]). Other methods make assumption that the estimated parameters are constant during the identification process. Control algorithms exploiting this type of parameter need to be updated on-line in order to increase their performances (Kenne et al. [2008]). In some bioprocess, the concept of elemental balances of some components in the bioreactor is used to obtain indirect measurements of various time-varying parameters. The reconstruction of the immeasurable state variables and of the time-varying parameters is carried out on the basis of Extended Kalman filter algorithm using the indirect measurements. But these algorithms are difficult to implement in practice, since very few sensors are able to provide reliable and on-line measurements of state variables (Lubenova [1999]). For discrete time systems, recursive algorithm design for joint state and parameter estimation can be found, but with the assumption of some canonical form for the system matrices (Guyader and Zhang [2003]). A natural idea for joint state and parameter estimation is to apply the Kalman filter to the extended system obtained by appending the unknown parameter into the state vector. However, the application of classical results requires uniform complete observability, which is difficult to chek for the extended system (Guyader and Zhang [2003]).

In the present paper, a systematic procedure is presented to deal with the state and parameter estimation for time-varying systems. It consists in transforming the original system into a polytopic linear model based on the sector nonlinearity approach and the convex polytopic transformation. This transformation has the major interest to represent exactly the system without any loss of informations since the considered nonlinearities are bounded (each parameter varies between two known values).

Up to our knowledge, this is the first contribution where the time-varying problem is treated is such a way.

The main advantage is to give a systematic procedure in order to rewrite the nonlinearities and establish the convergence conditions of the state and parameter estimation errors, which will be expressed in linear matrix inequalities (LMI) formulation using the Lyapunov method.

The paper is organized as follows. Section 2 introduces the polytopic structure for modelling and some preliminary results. It is followed by the representation of the nonlinear time-varying parameter by a polytopic structure in section 3. In section 4, observers for joint state and unknowm time-varying parameters estimation are implemented. In section 5, an extension for the case where a noise affects the output measurement with filter synthesis is applied in order to attenuate the influence of the parameter variation and the measurement noise on the state and parameter estimation errors. A numerical example and some simulation results are given in section 6. Conclusions are detailed in section 7.

2. PRELIMINARIES: POLYTOPIC LINEAR STRUCTURE FOR MODELING

The polytopic model may have different names, such as fuzzy model (Takagi-Sugeno model), multi-model, local model networks, ect. It allows the representation of the nonlinear behavior of systems by the interpolation of a set of linear submodels. Each submodel contributes to the global behavior of the nonlinear system through a weighting function $\mu_i(\xi(t))$ (Tanaka and Wang [2001]). The polytopic structure is given by

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(C_i x(t) + D_i u(t)) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the system state variable, $u(t) \in \mathbb{R}^{n_u}$ is the control input and $y(t) \in \mathbb{R}^m$ is the system output. $\xi(t) \in \mathbb{R}^q$ is the decision variable vector assumed to be measurable (as the system output) or known (as the system input). The weighting functions $\mu_i(\xi(t))$, also known as the scheduling functions, of the r submodels satisfy the convex sum property

$$\begin{cases} \sum_{i=1}^{r} \mu_i(\xi(t)) = 1 \\ 0 \leq \mu_i(\xi(t)) \leq 1, \quad i = 1, \ldots, r \end{cases} \tag{2}$$

In the remaining of the paper, the following lemma is used:

Lemma 1. Consider two matrices X and Y with appropriate dimensions, a time-varying matrix $\Delta(t)$ and a positive scalar ε. The following property is verified

$$X^T \Delta^T(t) Y + Y^T \Delta(t) X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y \tag{3}$$

for $\Delta^T(t)\Delta(t) \leq I$.

3. POLYTOPIC MODELLING OF TIME-VARYING PARAMETERS

The main contribution of this work is to jointly estimate the state variables and the time-varying parameters of a LPV system, using the PLM representation. For that, each time-varying parameter is rewritten under a particular form.

3.1 Scalar time varying parameter

Let us consider the time-varying linear system represented by equation (4)

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = Cx(t) \end{cases} \tag{4}$$

with

$$\begin{cases} A(t) = A_0 + \theta(t)A_1, \ \theta(t) \in [\underline{\theta}, \overline{\theta}] \\ B(t) = B_0 + \theta(t)B_1 \end{cases} \tag{5}$$

A_0, A_1, B_0 and B_1 are known matrices with suitable dimensions. $\theta(t) \in \mathbb{R}$ is a time-varying parameter, it is non measurable but bounded and affects both matrices A and B.

Using the so-called sector nonlinearity approach, the bounded parameter $\theta(t)$ can be rewritten as:

$$\theta(t) = \mu_1(\theta(t))\underline{\theta} + \mu_2(\theta(t))\overline{\theta} \tag{6}$$

with

$$\begin{cases} \mu_1(\theta(t)) = \dfrac{\overline{\theta} - \theta(t)}{\overline{\theta} - \underline{\theta}} \\ \mu_2(\theta(t)) = \dfrac{\theta(t) - \underline{\theta}}{\overline{\theta} - \underline{\theta}} \end{cases} \tag{7}$$

such that the weighting functions $\mu_1(\theta(t))$ and $\mu_2(\theta(t))$ verify the convex sum property (2) with $r = 2$.

Replacing (6) and (5) in equation (4), the initial linear system with the time-varying parameter $\theta(t)$ is now expressed as a PLM, such that :

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(\theta(t))(\overline{A}_i x(t) + \overline{B}_i u(t)) \tag{8}$$

with

$$\begin{cases} \overline{A}_1 = A_0 + \underline{\theta}\, A_1, \quad \overline{B}_1 = B_0 + \underline{\theta}\, B_1 \\ \overline{A}_2 = A_0 + \overline{\theta}\, A_1, \quad \overline{B}_2 = B_0 + \overline{\theta}\, B_1 \end{cases} \tag{9}$$

3.2 Extension to the vector case

Let us now consider the time-varying linear system represented by (10) with n parameters $\theta_i(t)$

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = Cx(t) \end{cases} \tag{10}$$

with

$$\begin{cases} A(t) = A + \sum_{j=1}^{n} \theta_j(t)\overline{A}_j \\ B(t) = B + \sum_{j=1}^{n} \theta_j(t)\overline{B}_j \end{cases} \tag{11}$$

Remark 1. In (11), it is supposed that the matrices $A(t)$ and $B(t)$ depend on the same parameters. If a parameter $\theta_j(t)$ does not affect $A(t)$ (resp. $B(t)$), then the corresponding matrix $\bar{A}_j$ (resp. $\bar{B}_j$) is null.

Each parameter $\theta_i(t)$ is expressed as:

$$\theta_i(t) = \mu_i^1(\theta_i(t))\underline{\theta}_i + \mu_i^2(\theta_i(t))\overline{\theta}_i \tag{12}$$

with the definitions:

$$\begin{cases} \mu_i^1(\theta_i(t)) = \dfrac{\overline{\theta}_i - \theta_i(t)}{\overline{\theta}_i - \underline{\theta}_i} \\ \mu_i^2(\theta_i(t)) = \dfrac{\theta_i(t) - \underline{\theta}_i}{\overline{\theta}_i - \underline{\theta}_i} \end{cases} \tag{13}$$

satisfying the constraint:

$$\mu_i^1(\theta_i(t)) + \mu_i^2(\theta_i(t)) = 1, \ \forall t, \ i = 1, \ldots, n$$

Finally, by replacing (11) and (12) in equation (10), we get:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{n} \sum_{j=1}^{2} \mu_i^j(\theta_i(t))(\overline{A}_i^j x(t) + \overline{B}_i^j u(t)) \\ y(t) = Cx(t) \end{cases} \tag{14}$$

$$\begin{array}{l} \overline{A}_i^1 = A_0 + \underline{\theta}_i\, A_i^1, \quad \overline{B}_i^1 = B_0 + \underline{\theta}_i\, B_i^1 \\ \overline{A}_i^2 = A_0 + \overline{\theta}_i\, A_i^1, \quad \overline{B}_i^2 = B_0 + \overline{\theta}_i\, B_i^1 \end{array} \tag{15}$$

4. STATE AND TIME-VARYING PARAMETER OBSERVER

Based on the obtained PLM, a simultaneous state and parameter observer may be designed and implemented. An $\mathscr{L}_2$ attenuation is proposed to minimize the effect of the time-varying parameters on the state estimation errors, since these parameters are unknown.

The state and parameter observer is taken as the following

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{n}\sum_{j=1}^{2}\mu_i^j(\hat{\theta}_i(t))(\overline{A}_i^j\hat{x}(t) + \overline{B}_i^j u(t)) + L_i^j(y(t) - \hat{y}(t)) \\ \dot{\hat{\theta}}(t) = \sum_{i=1}^{n}\sum_{j=1}^{2}\mu_i^j(\hat{\theta}_i(t))(K_i^j(y(t) - \hat{y}(t)) - \alpha_i^j\hat{\theta}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \quad (16)$$

where $L_i^j \in \mathbb{R}^{n_x \times m}$, $K_i^j \in \mathbb{R}^{n \times m}$ and $\alpha_i^j \in \mathbb{R}^{n \times n}$ are the gain matrices to be determined in such a way that simultaneously the observer state and parameter converge to the system state and parameter.

It may be noted here that the estimation problem is not trivial since the activation functions in the system (14) depend on $\theta(t)$, while those of the observer (16) depend on its estimate $\hat{\theta}(t)$.

Let us define the state estimation error $e_x(t)$ as

$$e_x(t) = x(t) - \hat{x}(t) \quad (17)$$

Its dynamics cannot be directly computed from (17) since the weighting functions depend on the unmeasurable variable $(\theta(t))$. That is why, based on the convex sum property of the weighting functions, the state equation (14) is rewritten as follows

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{n}\sum_{j=1}^{2}\mu_i^j(\hat{\theta}_i(t))\left(\overline{A}_i^j x(t) + \overline{B}_i^j u(t)\right) + \\ &\sum_{i=1}^{n}\sum_{j=1}^{2}\left(\mu_i^j(\theta_i(t)) - \mu_i^j(\hat{\theta}_i(t))\right)\left(\overline{A}_i^j x(t) + \overline{B}_i^j u(t)\right) \end{aligned} \quad (18)$$

This form allows a better comparison of $x(t)$ with $\hat{x}(t)$, since $\mu_i^j(\hat{\theta}_i(t))$ appears in the two expressions (16) and (18).

Let us define:

$$\begin{aligned} \Delta A_i(t) &= \sum_{j=1}^{2}(\mu_i^j(\theta_i(t)) - \mu_i^j(\hat{\theta}_i(t)))\overline{A}_i^j \\ &= \mathscr{A}_i \Sigma_{A_i}(t) E_A \\ \Delta B_i(t) &= \sum_{j=1}^{2}(\mu_i^j(\theta_i(t)) - \mu_i^j(\hat{\theta}_i(t)))\overline{B}_i^j \\ &= \mathscr{B}_i \Sigma_{B_i}(t) E_B \end{aligned} \quad (19)$$

with

$$\begin{aligned} &\mathscr{A}_i = \left[\, \overline{A}_i^1 \;\; \overline{A}_i^2 \,\right],\; \Sigma_{A_i}(t) = \begin{pmatrix} \delta_i^1(t) I_{n_x} & 0 \\ 0 & \delta_i^2(t) I_{n_x} \end{pmatrix}, \\ &\mathscr{B}_i = \left[\, \overline{B}_i^1 \;\; \overline{B}_i^2 \,\right],\; \Sigma_{B_i}(t) = \begin{pmatrix} \delta_i^1(t) I_{n_u} & 0 \\ 0 & \delta_i^2(t) I_{n_u} \end{pmatrix}, \\ &E_A = [\, I_{n_x} \;\; I_{n_x} \,]^T,\; E_B = [\, I_{n_u} \;\; I_{n_u} \,]^T \end{aligned} \quad (20)$$

Thanks to the property (2), for $i = 1,2$, it holds

$$\begin{cases} \delta_i^j(t) \;=\; \mu_i^j(\theta_i(t)) - \mu_i^j(\hat{\theta}_i(t)) \\ -1 \leq \delta_i^j(t) \leq 1 \end{cases} \quad (21)$$

which implies from definition (20)

$$\Sigma_{A_i}^T(t)\Sigma_{A_i}(t) \leq I, \quad \Sigma_{B_i}^T(t)\Sigma_{B_i}(t) \leq I \quad (22)$$

The system (18) is then written as an uncertain-like system given by (see Ichalal et al. [2009]):

$$\dot{x}(t) = \sum_{i=1}^{n}\sum_{j=1}^{2}\mu_i^j(\hat{\theta}_i(t))\left((\overline{A}_i^j + \Delta A_i(t))x(t) + (\overline{B}_i^j + \Delta B_i(t))u(t)\right) \quad (23)$$

From equations (23), (16) and (17), the dynamics of the state estimation error are given by

$$\begin{aligned} \dot{e}_x(t) = &\sum_{i=1}^{n}\sum_{j=1}^{2}\mu_i^j(\hat{\theta}_i(t)) \\ &\left((\overline{A}_i^j - L_i^j C)e_x(t) + \Delta A_i(t)x(t) + \Delta B_i(t)u(t)\right) \end{aligned} \quad (24)$$

Let us now define the parameter estimation error $e_\theta(t)$ as

$$e_\theta(t) = \theta(t) - \hat{\theta}(t) \quad (25)$$

From equation (16), the dynamics of this error is given by

$$\dot{e}_\theta(t) = \sum_{i=1}^{n}\sum_{j=1}^{2}\mu_i^j(\hat{\theta}_i(t))\left(\dot{\theta}(t) - K_i^j C e_x(t) + \alpha_i^j\theta(t) - \alpha_i^j e_\theta(t)\right) \quad (26)$$

Due to the coupling between the errors $e_\theta(t)$ and $e_x(t)$, it is convenient to consider the augmented vectors $e_a(t)$ and $\omega(t)$

$$e_a(t) = \begin{pmatrix} e_x(t) \\ e_\theta(t) \end{pmatrix},\; \omega(t) = \begin{pmatrix} x(t) \\ \theta(t) \\ \dot{\theta}(t) \\ u(t) \end{pmatrix} \quad (27)$$

in order to obtain the augmented system describing the state and parameter estimation errors.

$$\dot{e}_a(t) = \sum_{i=1}^{n}\sum_{j=1}^{2}\mu_i^j(\hat{\theta}_i(t))\left(\Phi_i^j e_a(t) + \Psi_i^j(t)\omega(t)\right) \quad (28)$$

with

$$\begin{aligned} \Phi_i^j &= \begin{pmatrix} \overline{A}_i^j - L_i^j C & 0 \\ -K_i^j C & -\alpha_i^j \end{pmatrix} \\ \Psi_i^j(t) &= \begin{pmatrix} \Delta A_i(t) & 0 & 0 & \Delta B_i(t) \\ 0 & \alpha_i^j & I & 0 \end{pmatrix} \end{aligned} \quad (29)$$

Our objective is to design the joint state and parameter observer with a minimal $\mathscr{L}_2$-gain of the transfer from $\omega(t)$ to $e_a(t)$. The computation of the observer gains is detailed in the next theorem.

Theorem 1. There exists a joint robust state and parameter observer (16) for a linear time-varying parameter system (10) with an $\mathscr{L}_2$-gain from $\omega(t)$ to $e_a(t)$ bounded by β if there exists $P_0 = P_0^T > 0$, $P_1 = P_1^T > 0$, $\beta > 0$, λ_{1i}, λ_{2i}, Γ_2^0, Γ_2^1, Γ_2^2, $\Gamma_2^3 > 0$, $\overline{\alpha}_i^j$, F_i^j and R_i^j solution of the optimization problem (30) under LMI constraints (32), for $i = 1,\ldots,n$ and $j = 1,2$:

$$\min_{P_0,P_1,R_i^j,F_i^j,\overline{\alpha}_i^j,\lambda_{1i},\lambda_{2i},\Gamma_2^0,\Gamma_2^1,\Gamma_2^2,\Gamma_2^3} \beta \quad (30)$$

$$\Gamma_2^k < \beta I,\ \text{for}\ k = 0,1,2,3 \quad (31)$$

The observer gains are given by

$$\begin{cases} L_i^j = P_0^{-1}R_i^j \\ K_i^j = P_1^{-1}F_i^j \\ \alpha_i^j = P_1^{-1}\overline{\alpha}_i^j \end{cases} \quad (33)$$

Proof 1. Let us consider the following quadratic Lyapunov function

$$V(e_a(t)) = e_a^T(t)Pe_a(t),\ P = P^T > 0 \quad (34)$$

its time derivative is given by

$$\begin{aligned} \dot{V}(t) = &\sum_{i=1}^{n}\sum_{j=1}^{2}\mu_i^j(\hat{\theta}_i(t))(e_a^T(t)((\Phi_i^j)^T P + P\Phi_i^j)e_a(t) \\ &+ e_a^T(t)P\Psi_i^j(t)\omega(t) + \omega^T(t)(\Psi_i^j)^T(t)Pe_a(t)) \end{aligned} \quad (35)$$

Considering equation (28), the goal of the observer design is to attenuate the effect of the input $\omega(t)$ on $e_a(t)$. So, in order

$$\begin{pmatrix} P_0\overline{A}_i^j + (\overline{A}_i^j)^T P_0 - R_i^j C - C^T (R_i^j)^T + I_{n_x} & -C^T (F_i^j)^T & 0 & 0 & 0 & 0 & P_0\mathscr{A}_i & P_0\mathscr{B}_i \\ * & -\overline{\alpha}_i^j - (\overline{\alpha}_i^j)^T + I_n & 0 & \overline{\alpha}_i^j & P_1 & 0 & 0 & 0 \\ * & * & -\Gamma_2^0 + \lambda_1 E_A^T E_A & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2^1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2^2 & 0 & 0 & 0 \\ * & * & * & * & * & -\Gamma_2^3 + \lambda_2 E_B^T E_B & 0 & 0 \\ * & * & * & * & * & * & -\lambda_{i1} I & 0 \\ * & * & * & * & * & * & 0 & -\lambda_{i2} I \end{pmatrix} < 0 \quad (32)$$

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to guarantee the stability of (28) and the boundedness of the transfer from $\omega(t)$ to $e_a(t)$, the following criterion is considered

$$\dot{V}(t) + e_a^T(t)e_a(t) - \omega^T(t)\Gamma_2\omega(t) < 0 \quad (36)$$

with

$$\Gamma_2 = \mathrm{diag}(\Gamma_2^k),\ \Gamma_2^k < \beta I,\ \text{for } k = 0,1,2,3 \quad (37)$$

such that Γ_2 allows to attenuate the transfer of some $\omega(t)$ components to $e_a(t)$ components.
From (35), (36) becomes:

$$\sum_{i=1}^{n}\sum_{j=1}^{2} \mu_i^j(\hat{\theta}_i(t)) \begin{pmatrix} e_a(t) \\ \omega(t) \end{pmatrix}^T \left(\begin{pmatrix} (\Phi_i^j)^T P + P\Phi_i^j + I & P\Psi_i^j(t) \\ (\Psi_i^j(t))^T(t)P & -\Gamma_2 \end{pmatrix} \right) \begin{pmatrix} e_a(t) \\ \omega(t) \end{pmatrix} < 0 \quad (38)$$

The main difficulty for satisfying (38) is the presence of time-varying terms. Then, the idea is to isolate and bound these terms. For that purpose, a block diagonal structure for the Lyapunov matrice P is considered:

$$P = \mathrm{diag}(P_0, P_1) \quad (39)$$

From (25), (37) and (39), (38) is explicited as

$$\sum_{i=1}^{n}\sum_{j=1}^{2} \mu_i^j(\hat{\theta}_i(t)) \left(Q_i^j + \mathscr{Q}_i(t) + \mathscr{Q}_i^T(t) \right) < 0 \quad (40)$$

with:

$$Q_i^j = \begin{pmatrix} (Q_i^j)_{11} & -C^T(K_i^j)^T P_1 & 0 & 0 & 0 & 0 \\ * & (Q_i^j)_{22} & 0 & P_1\alpha_i^j & P_1 & 0 \\ * & * & -\Gamma_2^0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2^1 & 0 & 0 \\ * & * & * & * & -\Gamma_2^2 & 0 \\ * & * & * & * & * & -\Gamma_2^3 \end{pmatrix} \quad (41)$$

$$\begin{aligned} (Q_i^j)_{11} &= P_0\overline{A}_i^j + (\overline{A}_i^j)^T P_0 - C^T(L_i^j)^T P_0 - P_0 L_i^j C + I_{n_x} \\ (Q_i^j)_{22} &= -P_1\alpha_i^j - (\alpha_i^j)^T P_1 + I \end{aligned} \quad (42)$$

Based on (19) and (22), the time-varying term of (40) can be expressed as:

$$\mathscr{Q}_i = \begin{pmatrix} P_0\mathscr{A}_i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \Sigma_A(t)E_{A_i} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} P_0\mathscr{B}_i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \Sigma_B(t)E_{B_i} \end{pmatrix} \quad (43)$$

Using Lemma 1, there exists positive scalars λ_{1i} and λ_{2i} such that

$$\mathscr{Q}_i(t) + \mathscr{Q}_i^T(t) < \begin{pmatrix} \mathscr{Q}_i^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{1i}E_A^T E_A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_{2i}E_B^T E_B \end{pmatrix} \quad (44)$$

with:

$$\mathscr{Q}_i^1 = \lambda_{1i}^{-1} P_0\mathscr{A}_i\mathscr{A}_i^T P_0 + \lambda_{2i}^{-1} P_0\mathscr{B}_i\mathscr{B}_i^T P_0 \quad (45)$$

From inequality (44) and with the variable changes (33), the LMI (32) implies (40) and then implies the $\mathscr{L}_2$-gain attenuation of the transfer from $\omega(t)$ to $e_a(t)$, which achieves the proof.

In order to improve the obtained results, the conditions (32) may be relaxed using the convex sum property (2). Since

$$\mu_i^2(t) = 1 - \mu_i^1(t) \quad (46)$$

(23) becomes:

$$\dot{x}(t) = \sum_{i=1}^{n}\sum_{j=1}^{2} \mu_i^1(\hat{\theta}_i(t)) \left((\overline{A}_i^j + \Delta A_i(t))x(t) + (\overline{B}_i^j + \Delta B_i(t))u(t) \right) \quad (47)$$

$$\begin{aligned} \Delta A_i(t) &= \delta_i^1(t)(\overline{A}_i^1 - \overline{A}_i^2) \\ \Delta B_i(t) &= \delta_i^1(t)(\overline{B}_i^1 - \overline{B}_i^2) \\ \delta_i^1(t) &= \mu_i^1(\theta(t)) - \mu_i^1(\hat{\theta}(t)) \end{aligned} \quad (48)$$

Based on the same developments as previously, the following theorem can be established:

Theorem 2. There exists a robust state and parameter observer (16) for the linear time-varying parameter system (10) with a bounded $\mathscr{L}_2$ gain β of the transfer from $\omega(t)$ to $e_a(t)$ ($\beta > 0$) if there exists $P_0 = P_0^T > 0$, $P_1 = P_1^T > 0$, $\beta > 0$, Λ_{1i}, Λ_{2i}, Γ_2^0, Γ_2^1, Γ_2^2, $\Gamma_2^3 > 0$, $\overline{\alpha}_i^j$, F_i^j and R_i^j solution of the optimization problem (49) under LMI constraints (51), for $i = 1,\ldots,n$ and $j = 1,2$:

$$\min_{P_0,P_1,R_i^j,F_i^j,\overline{\alpha}_i^j,\Lambda_{1i},\Lambda_{2i},\Gamma_2^0,\Gamma_2^1,\Gamma_2^2,\Gamma_2^3} \beta \quad (49)$$

$$\Gamma_2^k < \beta I \text{ for } k = 0,1,2,3 \quad (50)$$

where Λ_{1i} and Λ_{2i} are matrices (instead of scalars for the previous development) and where:

$$\begin{aligned} \Lambda_{Ai} &= (\overline{A}_i^1 - \overline{A}_i^2)^T \Lambda_{1i} (\overline{A}_i^1 - \overline{A}_i^2) \\ \Lambda_{Bi} &= (\overline{B}_i^1 - \overline{B}_i^2)^T \Lambda_{2i} (\overline{B}_i^1 - \overline{B}_i^2) \end{aligned} \quad (52)$$

The observer gains are still given by (33).

Proof 2. The proof is omitted for space limitation but is similar to the previous one.

5. NOISE MEASUREMENT AND FILTER SYNTHESIS

In order to prove the efficiency of our approach, we consider the presence of measurement noise. The system is then described by the following equations:

$$\begin{pmatrix} P_0\overline{A}_i^j + (\overline{A}_i^j)^T P_0 - R_i^j C - C^T(R_i^j)^T + I_{n_x} & -C^T(F_i^j)^T & 0 & 0 & 0 & 0 & P_0 & P_0 \\ * & -\overline{\alpha}_i^j - (\overline{\alpha}_i^j)^T + I_n & 0 & \overline{\alpha}_i^j & P_1 & 0 & 0 & 0 \\ * & * & -\Gamma_2^0 + \Lambda_{Ai} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2^1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2^2 & 0 & 0 & 0 \\ * & * & * & * & * & -\Gamma_2^3 + \Lambda_{Bi} & 0 & 0 \\ * & * & * & * & * & * & -\Lambda_{1i} & 0 \\ * & * & * & * & * & * & 0 & -\Lambda_{2i} \end{pmatrix} < 0 \quad (51)$$

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = Cx(t) + Gb(t) \end{cases} \quad (53)$$

where $b(t)$ is the noise measurement and matrices $A(t)$ and $B(t)$ have been already defined in (5).
The state and parameter observer is chosen as:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{n}\sum_{j=1}^{2} \mu_i^j(\hat{\theta}_i(t))(\overline{A}_i^j\hat{x}(t) + \overline{B}_i^j u(t)) + L_i^j(y(t) - \hat{y}(t)) \\ \dot{\hat{\theta}}(t) = \sum_{i=1}^{n}\sum_{j=1}^{2} \mu_i^j(\hat{\theta}_i(t))(K_i^j(y(t) - \hat{y}(t)) - \alpha_i^j\hat{\theta}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \quad (54)$$

Let us consider the augmented vectors $e_a(t)$ and $\omega(t)$

$$e_a(t) = \begin{pmatrix} e_x(t) \\ e_\theta(t) \end{pmatrix}, \; \omega(t) = \begin{pmatrix} x(t) \\ \theta(t) \\ \dot{\theta}(t) \\ u(t) \\ b(t) \end{pmatrix} \quad (55)$$

where $\omega(t)$ takes now into account the noise $b(t)$.
It follows:

$$\dot{e}_a(t) = \sum_{i=1}^{n}\sum_{j=1}^{2} \mu_i^j(\hat{\theta}_i(t))\left(\Phi_i^j e_a(t) + \Psi_i^j(t)\omega(t)\right) \quad (56)$$

with

$$\begin{aligned} \Phi_i^j &= \begin{pmatrix} \overline{A}_i^j - L_i^j C & 0 \\ -K_i^j C & -\alpha_i^j \end{pmatrix} \\ \Psi_i^j(t) &= \begin{pmatrix} \Delta A_i(t) & 0 & 0 & \Delta B_i(t) & -L_i^j G \\ 0 & \alpha_i^j & I & 0 & -K_i^j G \end{pmatrix} \end{aligned} \quad (57)$$

Our objective is to attenuate the effect of the parametric variation and the noise on the state and parameter estimations. The computation of the observer gains is detailed in the next theorem.

Theorem 3. There exists a robust state and parameter observer (54) for a linear time-varying parameters system (53) subject to noise measurement with a bounded $\mathscr{L}_2$ gain β of the transfer from $\omega(t)$ to $e_a(t)$ ($\beta > 0$) if there exists $P_0 = P_0^T > 0$, $P_1 = P_1^T > 0$, $\beta > 0$, Λ_{1i}, Λ_{2i}, Γ_2^0, Γ_2^1, Γ_2^2, $\Gamma_2^3 > 0$, $\Gamma_2^4 > 0$, $\overline{\alpha}_i^j$, F_i^j and R_i^j solution of the optimization problem (58) under LMI constraints (60), for $i = 1,\ldots,n$ and $j = 1,2$:

$$\min_{P_0,P_1,R_i^j,F_i^j,\overline{\alpha}_i^j,\Lambda_{1i},\Lambda_{2i},,\Gamma_2^0,\Gamma_2^1,\Gamma_2^2,\Gamma_2^3} \beta \quad (58)$$

$$\Gamma_2^k < \beta \text{ for } k = 0,1,2,3,4 \quad (59)$$

with:

$$\begin{aligned} \Lambda_{Ai} &= (\overline{A}_i^1 - \overline{A}_i^2)^T \Lambda_{1i}(\overline{A}_i^1 - \overline{A}_i^2) \\ \Lambda_{Bi} &= (\overline{B}_i^1 - \overline{B}_i^2)^T \Lambda_{2i}(\overline{B}_i^1 - \overline{B}_i^2) \end{aligned} \quad (61)$$

The observer gains are given by (33).

Proof 3. The proof for Theorem 3 is based on the same development as the two previous results.

6. NUMERICAL EXAMPLE

The proposed approach is illustrated by an academic example. Let consider the linear time-varying system defined by:

$$\dot{x}(t) = \left(A_0 + \theta_1(t)A_1^1 + \theta_2(t)A_2^1\right)x(t) + Bu(t) \quad (62)$$

$$A_0 = \begin{pmatrix} -0.3 & -1 & -0.3 \\ 0.1 & -2 & -0.5 \\ -0.1 & 0 & -0.1 \end{pmatrix}, \; A_1^1 = \begin{pmatrix} 0 & -1.1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_2^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1.1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0.5 \\ 0.25 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Parameters $\theta_1(t)$ and $\theta_2(t)$ vary between $[0,1]$.
In order to illustrate the influence of the time-varying parameters, fig. 1 represents the states $x_n(t)$ of the nominal system (without varying parameters) i.e. $\dot{x}_n(t) = A_0x_n(t) + Bu(t)$ and of the time-varying system states $x_v(t)$ given by (62). One can see the state deviation caused by the time-varying parameter. The system input is depicted in fig. 2.

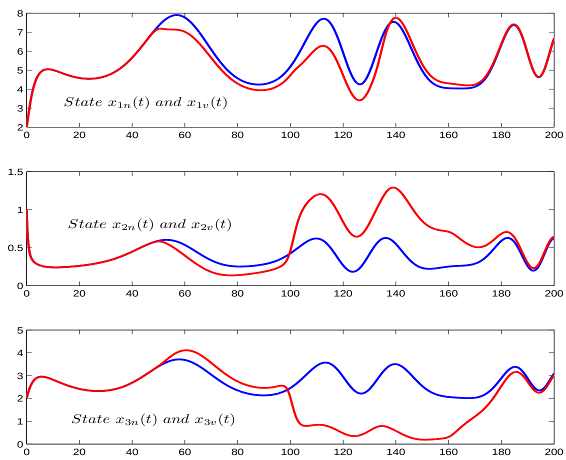

Figure 1. Nominal system (blue) -with parametric variation (red)

Considering a measurement noise defined by a normal distribution with zero mean and standard deviation 15% of the output magnitude affecting the system (53) with $G = I_2$, the observer gains are calculated from theorem 3. The actual and estimated states and parameters are depicted in fig. 3 and 4. The initial conditions for the system are taken $x_0 = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix}$ for the system and and $\hat{x}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ for the joint state and parameter observer.

$$\begin{pmatrix} P_0\overline{A}_i^j+(\overline{A}_i^j)^T P_0-R_i^j C-C^T(R_i^j)^T+I_{n_x} & -C^T(F_i^j)^T & 0 & 0 & 0 & 0 & -R_i^j G & P_0 & P_0 \\ * & -\overline{\alpha}_i^j-(\overline{\alpha}_i^j)^T+I_n & 0 & \overline{\alpha}_i^j & P_1 & 0 & -F_i^j G & 0 & 0 \\ * & * & -\Gamma_2^0+\Lambda_{Ai} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2^1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2^2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\Gamma_2^3+\Lambda_{Bi} & 0 & 0 & 0 \\ * & * & * & * & * & * & -\Gamma_2^4 & 0 & 0 \\ * & * & * & * & * & * & * & -\Lambda_{1i} & 0 \\ * & * & * & * & * & * & * & 0 & -\Lambda_{2i} \end{pmatrix} < 0 \qquad (60)$$

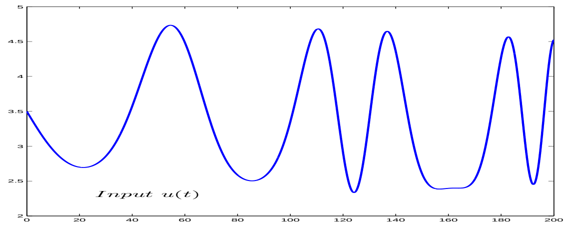

Figure 2. System input

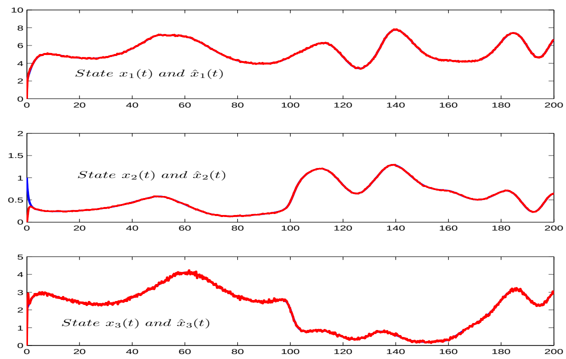

Figure 3. Actual and estimated states

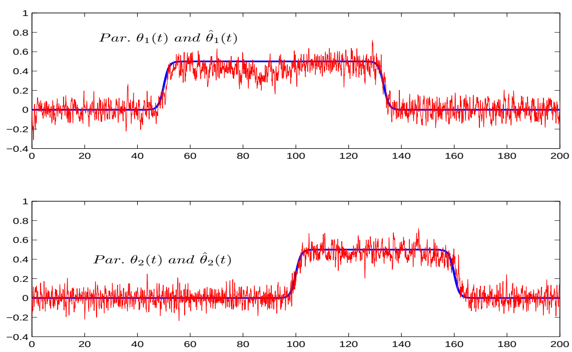

Figure 4. Time-varying parameters and their estimates

From the depicted figures, one can conclude that, even if the the measurements are affected by noises, acceptable state and time-varying parameters estimations are obtained.

7. CONCLUSION

In the present paper, a new systematic procedure is presented to deal with the state and parameter estimation for time-varying systems. It consists in transforming the original system into a Takagi-Sugeno model based on the sector nonlinearity approach and the convex polytopic transformation. This transformation has the major interest to exactly represent the system without any loss of information. Then, it is used for state and parameter observer design. The observer gains are given by LMI optimization in order to minimize the $\mathscr{L}_2$ gain from the inputs to the estimation errors. Relaxed conditions are also given based on the convex sum property of the weighting functions. The case with measurement noise is also studied.
As future work, the same approach may be applied for nonlinear system with time-varying parameters.

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