Observer design and fault tolerant control of Takagi-Sugeno nonlinear systems with unmeasurable premise variables[*]

Dalil Ichalal

Laboratoire d'Informatique, Biologie Intégrative et Systèmes Complexes (IBISC),
Univesité d'Evry Val d'Essonne, France

Benoît Marx, Didier Maquin, José Ragot

Centre de Recherche en Automatique de Nancy (CRAN),
Nancy-Université, France

1   Introduction

Dealing with real systems (electrical, mechanical, economic, chemical, ...) one is often urged to use nonlinear representations in order to accurately capture the whole behavior of the systems under study. The linear modeling of such systems only has a local validity, but it considerably eases the performance analysis and the controller supervision module designs. Even if a considerable amount of results have been established in the framework of linear systems, it is known that the linearity assumption is only valid around an operating point; consequently, the natural nonlinear behaviors of the system inevitably affect the performances of the control laws or supervision modules designed with the system linearity assumption. In order to enhance the system performances, it is necessary to take into account the nonlinear behaviors of the system from the modeling task to the control or diagnosis implementation. This can result in complex models to be dealt with, requiring heavy mathematical tools. Contrarily to what exists in the linear framework, the study of generic nonlinear models is too complex to lead to unified results. Consequently, many classes of nonlinear systems are studied with specific assumptions (LPV, Lipschitz, bilinear, ...). Recently, a lot of researches on stability, stabilization, observation and control of nonlinear systems are proposed by using Takagi-Sugeno (T-S) models [45]. This structure offers an efficient representation of nonlinear behaviors while remaining relatively simple compared to general nonlinear models $\dot{x} = f(x, u)$. Furthermore, some results developed in the

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linear framework can be extended to T-S models. So, the T-S modeling provides an alternative and attractive path to nonlinear system study.

Takagi-Sugeno modeling has proved its effectiveness in the study of nonlinear systems. T-S systems can be understood as polytopic systems, where the blending between the subsystems is time varying according to the so-called weighting functions. Thanks to the convexity of the blending and to the linearity of the subsystems defining the vertices, some results can be established for nonlinear systems, using tools borrowed from the linear theory. In that sense, the stability study and the controller/observer design of nonlinear systems is simplified. In [10, 20, 30, 19], the stability and stabilization tools inspired from the study of linear systems have been proposed. In [3, 34], the authors worked on the problem of state estimation and diagnosis of T-S systems. The proposed approaches in these last papers rely on the generalization of the classical observers (Luenberger Observer [32] and Unknown Input Observer (UIO) [13]) to the nonlinear domain. Most of the proposed results are formulated as optimization under linear matrix inequalities (LMI, [8]) constraints; these constraints are evaluated at the polytope vertices, defined by the submodels. The obtained conditions are only sufficient ones since the weighting functions are not explicitly taken into account thanks to the convexity of the potytope. This leads to some conservatism. Recently, some works were dedicated to the conservatism relaxation of the stability conditions. In [43], the Polya's theorem is used in order to reduce the conservatism related to the negativity of a polytope of matrix inequalities. In [30], the authors proposed a new approach for discrete time T-S systems, based on the evaluation of the Lyapunov function variation between two samples taken at times k and $k + m$ with $m > 1$.

Due to an increasing demand for higher performances, as well as for higher safety and reliability, fault detection and isolation (FDI) has been an active field of research over the past decades. Many techniques have been proposed especially for sensor and/or actuator faults with application to a wide range of engineering fields. Among them, model-based approaches to fault diagnosis for dynamic systems have received a lot of attention. As mentioned in [18], the purpose of FDI is to generate an alarm and identify the location of the fault as earlier as possible and different approaches have been proposed [11, 15, 14, 27]. A classification is given in [49] which decomposes diagnosis algorithms into quantitative model-based methods including observers, parity space and extended Kalman filter, and qualitative model-based methods which include graph approach, fault tree one, probabilistic approaches, etc. The quantitative approaches are used in order to achieve analytical redundancy. A well-known method is to generate residuals by comparing outputs of an observer and measured system outputs. For being usable, these residuals must be completely or as much as possible decoupled from unknown inputs and uncertainties. Depending on the quality of this decoupling, fixed or adaptive thresholds have to be determined in order to minimize false alarms [15, 4, 35]. The threshold can also be generated from statistical classifiers or neural networks approaches as proposed in [29]. These strategies, initially developed in the context of linear systems have been extended to the case of nonlinear ones represented by T-S models [42, 2, 18].

Once the fault have been detected and isolated, it is clearly interesting to minimize its effect on the concerned system and it is natural to be concerned with the development of a fault tolerant control (FTC) law. It consists in computing a new control law

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by taking into account the faults affecting the system in order to maintain acceptable performances and preserve the system stability even in faulty situations. The existing strategies are cast into two classes. The first one is the so-called passive fault tolerant control or robust control. In this approach, the faults are treated as uncertainties. Therefore, the control law is designed such that the closed-loop system is robust to the specified faults. Contrarily to the passive FTC, active FTC requires a FDI block to detect, isolate and estimate the faults. The information issued from the FDI block are used by the FTC module to reconfigure the control law in order to compensate the faults and ensure acceptable system performances. Active fault tolerant control has been developed essentially for linear systems [16, 44, 38, 36] and descriptor linear systems [33]. As explained before, it should be preferable to address FTC for nonlinear models. One can cite some works in FTC field for nonlinear systems, for example in [17], the authors proposed a method of actuator fault tolerant control for nonlinear descriptor systems with Lipschitz nonlinearities. Once again T-S model is a mean to extend some results in FTC to the nonlinear framework: [39] proposed a solution for FTC of T-S systems only requiring the fault isolation. The controller is driven by a bank of observers and a switching system is designed to switch from one controller to another, according to a decision logic based on residual signals.

In the present chapter, two main problems will be addressed: the first one is observer design and the second one is observer-based FTC, both for nonlinear systems described by T-S models with unmeasurable premise variables (UPV).

Indeed, the T-S models can be cast into two main classes depending whether the premise variables are measurable or not. It is clear that the choice of measurable premise variables eases the extension of the methods already developed for linear systems. The problems of state estimation and diagnosis of nonlinear systems using T-S model approach have been addressed with different methods, but most of the published works have considered T-S models with measurable premise variables [2, 42, 34, 3]. These problems become harder when the premise variables are not measurable. However, T-S systems with UPV are very useful both for the exact representation of nonlinear behaviors by T-S model and for observer based diagnosis for sensor/actuator fault detection and isolation. The sector nonlinearity transformation leads to T-S model with weighting functions depending on the state of the system which is partially or completely unmeasurable. Indeed in this case, the use of measurable premise variables requires to develop two different models. The first one uses the input $u(t)$ as a premise variable and allows to detect and isolate sensor faults. The second one, using the output $y(t)$ of the system as a premise variable, is dedicated to the detection and isolation of actuator faults. Diagnosis based on a single T-S model with unmeasurable premise variables allows to detect and isolate both actuator and sensor faults using observer banks with only one T-S model. Furthermore, the T-S models with unmeasurable premise variables may represent a larger class of nonlinear systems compared to the T-S model with measurable premise variables [50]. Only few works are devoted to the case of unmeasurable premise variables: in [41, 6, 7], the authors proposed the Thau-Luenberger observer which is an extension of the classical Luenberger observer and, in [50], a filter estimating the state and minimizing the effect of disturbances was proposed. Recently, other approaches for observer design, fault diagnosis and fault tolerant control, have been proposed for this class of systems in [23, 22, 21, 25, 26] based on Lipschitz

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property, on the mean value theorem or on the modeling of estimation error as model uncertainties to be robust to.

The overview of this chapter is as follows. A short introduction to T-S systems is presented in section 2 and some useful results are detailed in section 3. The section 4 is devoted to the design of an observer for continuous time T-S systems with unmeasurable premise variables. Firstly, the state estimation error dynamics are obtained as a T-S perturbed system where the considered perturbation depends on the system state and its estimate. Thanks to the mean value theorem (MVT), this perturbed T-S system with UPV is rewritten as an autonomous one. Lyapunov stability analysis is then used to derive sufficient existence conditions of the observer. These conditions are expressed in terms of LMIs, which can easily be solved by dedicated softwares (LMI toolbox of Matlab, YALMIP [31],...). A numerical example illustrates the proposed observer design. The second main result of this chapter, exposed in section 5, is dedicated to the design of a new active fault tolerant control design for T-S systems with UPV. Both additive actuator and sensor faults can be handled. In the proposed approach the objective is twofold: not only the occurring faults are compensated, but trajectory tracking is also ensured even in the faulty case. The reference state trajectory is a user defined model. As illustrated in the figure 1, an observer provides fault and state estimations, required for the control law reconfiguration. The stability of the system with the proposed fault tolerant control is studied via Lyapunov approach. The obtained conditions are formulated in terms of linear matrix inequalities. In section 6, the control of the lateral dynamics of a vehicle illustrates the approach.
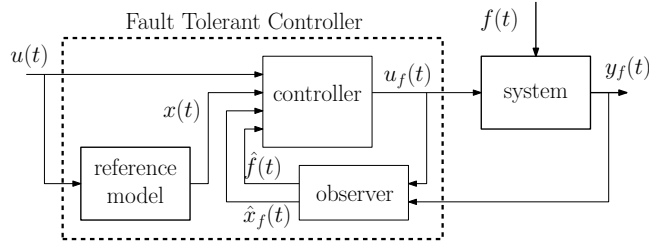

Figure 1: Fault tolerant control scheme

2 Takagi-Sugeno structure for modeling

Consider a nonlinear system described by

$$\begin{cases} \dot{x}(t) = f_x(x(t), u(t)) \\ y(t) = f_y(x(t)) \end{cases} \tag{1}$$

The T-S modeling allows to represent the behavior of the nonlinear system (1) by the interpolation of a set of linear sub-models. Each sub-model contributes to the global behavior of the nonlinear system through a weighting function $\mu_i(\xi(t))$. The T-S

structure is given by

$$
\begin{cases}
\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\
y(t) = \sum_{i=1}^{r} \mu_i(\xi(t)) C_i x(t)
\end{cases}
\tag{2}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$ represents the output vector. $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{p \times n}$ and $D_i \in \mathbb{R}^{p \times m}$ are known matrices. The weighting functions $\mu_i(\xi(t))$ depend on the variables $\xi(t)$ which can be measurable (as the input or the output of the system) or non measurable variables (as the state of the system). These functions verify the so-called convex sum property

$$
\begin{cases}
\sum_{i=1}^{r} \mu_i(\xi(t)) = 1 \\
0 \leq \mu_i(\xi(t)) \leq 1 \quad \forall i \in \{1, 2, ..., r\}
\end{cases}
\tag{3}
$$

Obtaining a T-S model (2) from (1), which reduces to obtain the matrices A_i, B_i C_i and the weighting functions μ_i, can be performed from different methods such as linearizing the system (1) around some operating points and using adequate weighting functions. It can also be obtained by black-box approaches allowing to identify the parameters of the model from input-output data. Finally, the most interesting and important way to obtain a T-S model in the form (2) is the well-known sector nonlinearity transformation [47, 37]. Indeed, this transformation allows to obtain an exact T-S representation of (1) with no information loss on a compact set of the state space.

Thanks to the convex sum property of the weighting functions (3), it is possible to generalize some tools developed in the linear domain to the nonlinear systems. The representation (2) is very interesting in the sense that it simplifies the stability studies of nonlinear systems and the design of control laws and observers. In [10, 20, 30], the stability and stabilization tools are inspired from the study of linear systems. In [3, 34], the authors worked on the problem of state estimation and diagnosis of T-S systems. The proposed approaches in these last papers rely on the generalization of the classical observers (Luenberger Observer [32] and Unknown Input Observer (UIO) [13]) to the nonlinear domain.

3 Useful results and notations

In the remaining of the paper, the following notations will be used.

Notation 1. *For any square matrix X, $\mathbb{S}$ is defined by $\mathbb{S}(X) = X + X^T$. The notation $X = diag(X_1, X_2, \ldots, X_n)$ defines X as a block diagonal matrix where the square matrices X_i are the entries on the diagonal of X. In a block matrix, the symbol $*$ denotes the terms induced by symmetry. In order to shorten the summation in the definitions of T-S*

systems, the following notations are defined

$$X_\mu(t) = \sum_{i=1}^{r} \mu_i(x(t))X_i \qquad\qquad X_{\hat{\mu}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))X_i \qquad (4)$$

$$X_{\mu_f}(t) = \sum_{i=1}^{r} \mu_i(x_f(t))X_i \qquad\qquad X_{\hat{\mu}_f}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}_f(t))X_i \qquad (5)$$

$$X_{\mu\mu}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(x(t))\mu_j(x(t))X_i \qquad X_{\mu_f\hat{\mu}_f}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(x_f(t))\mu_j(\hat{x}_f(t))X_i \qquad (6)$$

Lemma 1. *Consider two matrices X and Y with appropriate dimensions and Ω a positive definite matrix. The following property is verified*

$$X^T Y + Y^T X \leq X^T \Omega X + Y^T \Omega^{-1} Y \qquad (7)$$

Proof. For any matrices X, Y and $\Omega > 0$, it obviously holds

$$(\Omega^{1/2}X - \Omega^{-1/2}Y)^T(\Omega^{1/2}X - \Omega^{-1/2}Y) \geq 0 \qquad (8)$$

Developing the previous inequality (7) is obtained. □

Lemma 2. *(Congruence) [8] Let two matrices P and Q, if P is positive definite and if Q is a full column rank matrix, than the matrix QPQ^T is positive definite.*

Lemma 3. *(Mean value theorem) Consider $g(z) : \mathbb{R}^n \to \mathbb{R}$. Let $a, b \in \mathbb{R}^n$. If $g(z)$ is differentiable on $[a,b]$ then there exists a constant vector $\tilde{z} \in \mathbb{R}^n$ satisfying $\tilde{z} \in]a,b[$ (i.e. $\tilde{z}_i \in]a_i, b_i[$, for $i = 1, \ldots, n$), such that*

$$g(a) - g(b) = \frac{\partial g}{\partial z}(\tilde{z})(a - b) \qquad (9)$$

Lemma 4. *(Sector nonlinearity approach) [47, 37] Any nonlinear function $g(z)$ satisfying*

$$\underline{g} \leq g(z) \leq \overline{g}, \ \forall z \qquad (10)$$

can be written as

$$g(z) = \mu_1(z)\underline{g} + \mu_2(z)\overline{g} \qquad (11)$$

where

$$\mu_1(z) = \frac{\overline{g} - g(z)}{\overline{g} - \underline{g}} \qquad \mu_2(z) = \frac{g(z) - \underline{g}}{\overline{g} - \underline{g}} \qquad (12)$$

and the functions $\mu_i(z)$ satisfy the convex sum property i.e. $\mu_1(z) + \mu_2(z) = 1$ and $0 \leq \mu_i(z) \leq 1$, $\forall z$.

4 Observer design for T-S systems with unmeasurable premise variables

In this section, the design of observer for T-S systems with unmeasurable variables is proposed. The main idea of the presented approach is based on the Mean Value Theorem (MVT) allowing to represent the state estimation error as an autonomous T-S system. Applying Lyapunov stability analysis, sufficient existence conditions of the observer are derived and formulated in LMIs that can be solved with dedicated softwares. Using the notations introduced previously, the T-S system to be observed is defined by

$$\begin{cases} \dot{x}(t) = A_\mu(t)x(t) + B_\mu(t)u(t) \\ y(t) = Cx(t) \end{cases} \tag{13}$$

It can be noted that the state variables often correspond to physical growths that are measured by dedicated sensors. As a consequence it is not so restrictive to consider a linear measurement equation. The observer is given by

$$\begin{cases} \dot{\hat{x}}(t) = A_{\hat{\mu}}(t)\hat{x}(t) + B_{\hat{\mu}}(t)u(t) + L_{\hat{\mu}}(t)(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{14}$$

where the gains L_i are sought in order to ensure the asymptotic convergence of the state estimation error towards zero. The state estimation error $e(t) = x(t) - \hat{x}(t)$ is governed by the following differential equation

$$\dot{e}(t) = g(z(t)) - g(\hat{z}(t)) - L_{\hat{\mu}}(t)Ce(t) \tag{15}$$

where $z^T(t) = \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix}$, $\hat{z}^T(t) = \begin{bmatrix} \hat{x}^T(t) & u^T(t) \end{bmatrix}$ and

$$g(z(t)) = A_\mu(t)x(t) + B_\mu(t)u(t), \ \ g(\hat{z}(t)) = A_{\hat{\mu}}(t)\hat{x}(t) + B_{\hat{\mu}}(t)u(t)$$

The function $g(z(t)) : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is assumed to be Lipschitz continuous. Note that the stability analysis of (15) cannot be directly achieved with the help of the tools developed for T-S systems with measurable premise variables. Indeed, the fact that the premise variables are the state variables leads to a more complex form of the state estimation error (see equation (15)). The key point of the proposed observer design is to obtain a suitable form of the state estimation error in order to re-use the tools proposed for stability and relaxed stability analysis of T-S systems with measurable premise variables.

Let us denote $e_s(i)$ the vector of $\mathbb{R}^{s \times 1}$ with all entries being null, except the ith being equal to 1 as given below

$$e_s(i) = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & & i-1 & i & i+1 & & s \end{pmatrix}^T \tag{16}$$

The function $g(z(t))$, with entries $g_i(z(t))$, can be written as follows

$$g(z(t)) = \sum_{i=1}^{n} e_n(i)g_i(z(t)) \tag{17}$$

Since the function g is differentiable on $[z(t) \ \hat{z}(t)]$, applying the lemma 3, there exists n vectors $\tilde{z}^i(t) \in \mathbb{R}^{n+m}$, satisfying $\tilde{z}^i(t) \in]z(t), \ \hat{z}(t)[$ (i.e. $\tilde{z}^i_j(t) \in]z_j(t), \hat{z}_j(t)[$, for $j = 1,\dots,n+m$), such that

$$g(z(t)) - g(\hat{z}(t)) = \sum_{i=1}^{n} \sum_{j=1}^{n+m} e_n(i)e_{n+m}^T(j)\frac{\partial g_i}{\partial z_j}(\tilde{z}^i(t))(z(t) - \hat{z}(t)) \tag{18}$$

Due to the definition of $z(t)$, one can note that $z_j(t) - \hat{z}_j(t) = 0$, for $j = n+1,\dots,n+m$. From (18) and (15), it then follows

$$\dot{e}(t) = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} e_n(i)e_n^T(j)\frac{\partial g_i}{\partial x_j}(\tilde{z}^i(t)) - L_{\hat{\mu}}(t)C \right) e(t) \tag{19}$$

Since the function g is Lipschitz, its derivatives are bounded

$$\underline{g_{ij}} \le \frac{\partial g_i}{\partial x_j}(\tilde{z}^i(t)) \le \overline{g_{ij}} \tag{20}$$

Using the lemma 4, each derivative can be written as

$$\frac{\partial g_i}{\partial x_j}(\tilde{z}^i(t)) = \mu_1^{ij}(z(t))\underline{g_{ij}} + \mu_2^{ij}(z(t))\overline{g_{ij}} \tag{21}$$

where the functions μ_1^{ij} and μ_2^{ij} satisfy the convex sum property. Factorizing these weighting functions and using the sector nonlinearity approach, one obtains the following global T-S model [47, 37]

$$\dot{e}(t) = \sum_{i=1}^{q} \sum_{j=1}^{r} h_i(z(t))\mu_j(\hat{x}(t))\left(\mathcal{A}_i - L_jC\right)e(t) \tag{22}$$

$$= \left(\mathcal{A}_h(t) - L_{\hat{\mu}}(t)C\right)e(t) \tag{23}$$

where $q \le 2^{n^2}$. One can note that the functions $h_i(z(t))$ are defined by the products of some functions $\mu_1^{ij}(z(t))$ and/or $\mu_2^{ij}(z(t))$ and the entries of $\mathcal{A}_i$ are given by the upper or lower bounds of the derivatives of g in (20). An illustration of this transformation is provided in the beginning of the example in the following subsection.

Based on this transformation, the system describing the state estimation error is an autonomous T-S system with UPV. The gains L_j of the observer are determined in order to stabilize the system (23). They are obtained by solving the LMI problem given in the theorem 1.

Theorem 1. *The state estimation error asymptotically converges toward zero if there exist a symmetric positive definite matrix $P \in R^{n \times n}$ and matrices $M_j \in R^{n \times p}$ such that the following linear matrix inequalities hold $\forall i = 1,\dots,q, \ j = 1,\dots,r$*

$$\mathcal{A}_i^T P + P\mathcal{A}_i - M_jC - C^T M_j^T < 0 \tag{24}$$

The gains of the observer are derived from

$$L_j = P^{-1}M_j \tag{25}$$

Proof. Considering Lyapunov function candidate

$$V(e(t)) = e^T(t)Pe(t), \quad P = P^T > 0 \tag{26}$$

and equation (23), it is straightforward to obtain

$$\dot{V}(e(t)) = \sum_{i=1}^{q} \sum_{j=1}^{r} h_i(z(t))\mu_j(\hat{x}(t))e^T(t)\left((\mathcal{A}_i - L_jC)^T P + P(\mathcal{A}_i - L_jC)\right)e(t) \tag{27}$$

From (3), the weighting functions are nonnegative. With the change of variables $M_j = PL_j$, if the LMIs (24) hold, the time derivative of the Lyapunov function (27) is negative and consequently the state estimation error asymptotically converges towards zero. $\square$

4.1 Simulation example: Lorenz system

In this example, the proposed method is applied in order to construct an observer which allows to estimate the states of the chaotic Lorenz system given by

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \end{pmatrix} \\ y(t) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x(t) + \omega(t) \end{cases} \tag{28}$$

The output of the system is noised by the random signal $\omega(t)$ which is bounded by 3.5. It is known that all the trajectories of this system come to a ball of finite radius (bounded trajectories). From this information, an exact T-S model can be obtained for all the state space of the nonlinear system. The system (28) can be written as a quasi-LPV (Linear Parameter Varying) system as follows

$$\dot{x}(t) = g(x_1(t))x(t) \tag{29}$$

with

$$g(x_1(t)) = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_1(t) \\ 0 & x_1(t) & -\frac{8}{3} \end{pmatrix} \tag{30}$$

Let us chose $x_1(t)$ as the premise variable. Knowing that $x_1(t)$ is bounded, we have $\xi_{min} \leq x_1(t) \leq \xi_{max}$. From lemma 4, the function $g(x_1(t))$ can then be written as

$$g(x_1(t)) = \mu_1(x(t))A_1 + \mu_2(x(t))A_2 \tag{31}$$

where the weighting functions in (31) are defined by

$$\mu_1(x(t)) = \frac{x_1(t) - \xi_{min}}{\xi_{max} - \xi_{min}}, \quad \mu_2(x(t)) = \frac{\xi_{max} - x_1(t)}{\xi_{max} - \xi_{min}} \tag{32}$$

where $\xi_{min} = -30$ and $\xi_{max} = 30$ respectively denote the lower and upper bounds of $x_1(t)$. Moreover, the matrices A_i in (31), defining the T-S model, are given by

$$A_1 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & -30 \\ 0 & 30 & -\frac{8}{3} \end{pmatrix}, \quad A_2 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 30 \\ 0 & -30 & -\frac{8}{3} \end{pmatrix}$$

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Since the weighting functions depend on the unmeasured state $x_1(t)$, it is impossible to construct an observer with the classical approaches dedicated to T-S models with known premise variables. The gains of the observer that satisfy the conditions expressed in the theorem 1 are given by

$$L_1 = L_2 = \begin{pmatrix} 10.2971 \\ 62.2607 \\ 2.1562 \end{pmatrix}$$

That allows to construct the observer (14). The resulting estimated states are depicted in the figure 2.
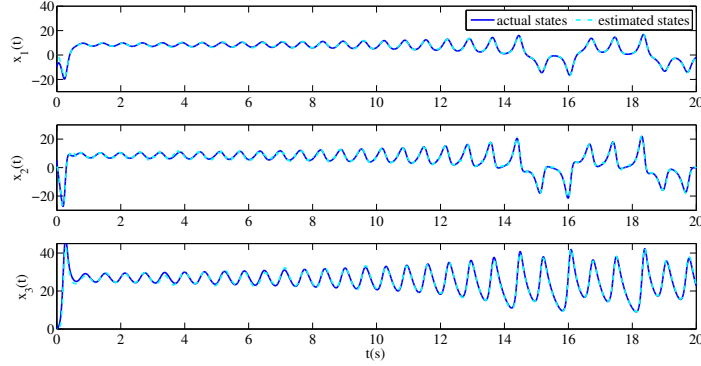

Figure 2: States $x(t)$ and estimated states $\hat{x}(t)$

Remark 1. *One can note that different quasi-LPV forms can be obtained from* (28). *For instance, the function $g(x(t))$ may not only depend on the first component of the system state, but on the two lasts (when factorizing the terms $x_1(t)x_3(t)$ and $x_1(t)x_2(t)$). This would result in two nonlinearities in $g(x)$ leading to four submodels and thus increasing the number of LMIs to be solved. Choice criteria between the different possible quasi-LPV forms are given in [37].*

Remark 2. *Note that the Lorenz system can be written as a Lipschitz model $\dot{x}(t) = Ax(t) + g(x(t))$. Many approaches have been proposed in literature to construct observers for such a system [12, 51, 1]. The main problem in these approaches is the value of the Lipschitz constant of $g(x(t))$. Indeed, if this constant is greater than an admissible value, there is no solution for LMIs. As discussed in [24], the T-S approach can overcome this limitation. Describing the Lipschitz model as a T-S one allows to obtain the gains of the state observer, even for large values of the Lipschitz constant. Moreover, if no solution exists to the LMI existence conditions of the observer for T-S model, it is possible to take benefits from relaxed conditions proposed in [48, 46, 43, 30]. In the proposed example, the classical approaches fail to provide a solution but after transformation, a solution is obtained.*

5 Fault tolerant control of T-S systems with UPV

Let us consider the T-S reference model without faults described by (33)

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{33}$$

The system with the fault $f(t) \in \mathbb{R}^{n_f}$ is described by the following T-S model with unmeasurable premise variables

$$\begin{cases} \dot{x}_f(t) = \sum_{i=1}^{r} \mu_i(x_f(t))(A_i x_f(t) + B_i u_f(t) + G_i f(t)) \\ y_f(t) = Cx_f(t) + Rf(t) \end{cases} \tag{34}$$

The goal is to design the control law $u_f(t)$ such that the system state $x_f(t)$ converges toward the reference state $x(t)$ given by the reference model (33), as illustrated by the figure 1.

The following structure is proposed for the fault tolerant control law

$$u_f(t) = \sum_{i=1}^{r} \mu_i(\hat{x}_f(t))\left(-K_{2i}\hat{f}(t) + K_{1i}(x(t) - \hat{x}_f(t)) + u(t)\right) \tag{35}$$

Analyzing the structure of $u_f(t)$ given in (35), the state and fault estimates, namely $\hat{x}_f(t)$ and $\hat{f}(t)$, are required, whereas the reference system state $x(t)$ is known and thus can be used.

There is no loss of generality to assume that the reference and faulty systems (33) and (34) are stable, compared to assume that they are stabilizable. If the systems (33) and (34) are stabilizable, there exists a stabilizing control law $u_{stab}(t)$. In this case, the control input $u(t)$ in (33) and (35) can be substituted by $u(t) + u_{stab}(t)$ and the matrices $A_\mu(t)$ and $A_{\mu_f}(t)$ can be replaced by their corresponding matrices in closed-loop.

The faulty state and fault estimates are provided by the following Proportional-Integral (PI) observer, which can be viewed as an extension to T-S systems of the well known linear PIO [5, 9, 28]

$$\begin{cases} \dot{\hat{x}}_f(t) = \sum_{i=1}^{r} \mu_i(\hat{x}_f(t))(A_i \hat{x}_f(t) + B_i u_f(t) + G_i \hat{f}(t) + H_{1i}(y_f(t) - \hat{y}_f(t))) \\ \dot{\hat{f}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}_f(t))H_{2i}(y_f(t) - \hat{y}_f(t)) \\ \hat{y}_f(t) = C\hat{x}_f(t) + R\hat{f}(t) \end{cases} \tag{36}$$

where the gains H_{1i} and H_{2i} are to be determined in order to minimize the faulty state and fault estimation errors.

The output error between the system (34) and the observer (36) is given by

$$y_f(t) - \hat{y}_f(t) = \overline{C}e_a(t) \tag{37}$$

where

$$\overline{C} = \begin{pmatrix} C & R \end{pmatrix}, \quad x_a(t) = \begin{pmatrix} x_f(t) \\ f(t) \end{pmatrix}, \quad \hat{x}_a(t) = \begin{pmatrix} \hat{x}_f(t) \\ \hat{f}(t) \end{pmatrix}, \quad e_a(t) = x_a(t) - \hat{x}_a(t) \quad (38)$$

The dynamic of the trajectory tracking error $e(t) = x(t) - x_f(t)$ obeys to the differential equation

$$\dot{e}(t) = A_\mu(t)x(t) + B_\mu(t)u(t) - (A_{\mu_f}(t)x_f(t) + B_{\mu_f}(t)u_f(t) + G_{\mu_f}(t)f(t)) \quad (39)$$

Taking into account the definition (35), (39) leads to

$$\dot{e}(t) = A_{\mu_f}(t)e(t) - G_{\mu_f}(t)f(t) - B_{\mu_f}(t)K_{2\hat{\mu}_f}(t)\hat{f}(t) - B_{\mu_f}(t)K_{1\hat{\mu}_f}(t)(x_f(t) - \hat{x}_f(t)) + \overline{\Delta}_1(t) \quad (40)$$

$$= (A_{\mu_f}(t) - B_{\mu_f}(t)K_{1\hat{\mu}_f}(t))e(t) - B_{\mu_f}(t)K_{\hat{\mu}_f}(t)e_a(t) + \overline{\Delta}_1(t) \quad (41)$$

where K_j and $\overline{\Delta}_1(t)$ are defined by

$$K_j = \begin{pmatrix} K_{1j} & K_{2j} \end{pmatrix} \quad (42)$$

$$\overline{\Delta}_1(t) = \Delta_1(t) + (B_{\mu_f}(t)K_{2\hat{\mu}_f}(t) - G_{\mu_f}(t))f(t) \quad (43)$$

$$\Delta_1(t) = \sum_{i=1}^{r} (\mu_i(x(t)) - \mu_i(x_f(t)))(A_i x(t) + B_i u(t)) \quad (44)$$

Note that the perturbation term $\Delta_1(t)$ is bounded since it is assumed that the system (33) is stable and that its input is bounded.

The dynamic of the fault estimation error is given by

$$\dot{e}_f(t) = \dot{f}(t) - \dot{\hat{f}}(t) \quad (45)$$

$$= \dot{f}(t) - H_{2\hat{\mu}_f}(t)\overline{C}e_a(t) \quad (46)$$

In order to analyze the evolution of the errors, two cases are considered : in the first one the faults are supposed to be constant ; in the second one they are assumed to be under a polynomial form with respect to the time variable.

5.1 Constant faults

In this first approach, it is supposed that $\dot{f}(t) = 0$ and, with definition (38), the system (34) becomes

$$\begin{cases} \dot{x}_a(t) = \sum_{i=1}^{r} \mu_i(x_f(t)) \left(\overline{A}_i x_a(t) + \overline{B}_i u_f(t) \right) \\ y_f(t) = \overline{C}x_a(t) \end{cases} \quad (47)$$

where $\overline{C}$ is already defined and

$$\overline{A}_i = \begin{pmatrix} A_i & G_i \\ 0 & 0 \end{pmatrix}, \quad \overline{B}_i = \begin{pmatrix} B_i \\ 0 \end{pmatrix} \quad (48)$$

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In a similar way, the observer (36) can be written using the augmented state vector $\hat{x}_a(t)$. The state and fault estimation errors between the system (47) and the observer (36), defined by $e_a(t) = x_a(t) - \hat{x}_a(t)$ are ruled by

$$\dot{e}_a(t) = \sum_{i=1}^{r} \mu_i(\hat{x}_f(t)) \left((\bar{A}_i - H_i\bar{C})e_a(t) + \Gamma\bar{\Delta}_2(t) \right) \tag{49}$$

where $\Gamma^T = \begin{pmatrix} I_n & 0 \end{pmatrix}$, $H_i^T = \begin{pmatrix} H_{1i}^T & H_{2i}^T \end{pmatrix}$ and

$$\bar{\Delta}_2(t) = \Delta_2(t) + \sum_{i=1}^{r} (\mu_i(x_f(t)) - \mu_i(\hat{x}_f(t)))G_i f(t) \tag{50}$$

$$\Delta_2(t) = \sum_{i=1}^{r} (\mu_i(x_f(t)) - \mu_i(\hat{x}_f(t)))(A_i x_f(t) + B_i u_f(t)) \tag{51}$$

Due to the convex sum property, $\bar{\Delta}_2(t)$ can also be written as

$$\bar{\Delta}_2(t) = \Delta_2(t) + \sum_{i=1}^{r} \sum_{j=1}^{r} (\mu_i(x_f(t))\mu_j(\hat{x}_f(t)))(G_i - G_j)f(t) \tag{52}$$

As explained above, concerning the term $\Delta_1(t)$ defined by (44), the perturbation $\Delta_2(t)$ is bounded. From (41) and (49), the concatenation of the state tracking trajectory error, the state and fault estimation errors allows to write the following augmented system

$$\dot{\tilde{e}}(t) = \tilde{A}_{\mu_f\hat{\mu}_f}(t)\tilde{e}(t) + \tilde{\Gamma}_{\mu_f\hat{\mu}_f}(t)\Delta(t) \tag{53}$$

where K_j is defined by (42) and

$$\tilde{e}(t) = \begin{pmatrix} x(t) - x_f(t) \\ x_f(t) - \hat{x}_f(t) \\ f(t) - \hat{f}(t) \end{pmatrix}, \quad \Delta(t) = \begin{pmatrix} \Delta_1(t) \\ \Delta_2(t) \\ f(t) \end{pmatrix}, \quad \tilde{\Gamma}_{ij} = \begin{pmatrix} \Gamma_{1ij} \\ \Gamma_{2ij} \end{pmatrix}, \quad \tilde{A}_{ij} = \begin{pmatrix} A_i - B_i K_{1j} & -B_i K_j \\ 0 & \bar{A}_i - H_i\bar{C} \end{pmatrix}$$

$$\tag{54}$$

$$\Gamma_{1ij} = \begin{pmatrix} I_n & 0 & B_i K_{2j} - G_i \end{pmatrix}, \quad \Gamma_{2ij} = \begin{pmatrix} 0 & I_n & G_i - G_j \\ 0 & 0 & 0 \end{pmatrix}, \tag{55}$$

From now on, the term $\Delta(t)$ gathering the fault and the mismatches coming from the unmeasured premise variables (44)-(51), is considered as an external disturbance, which effect is to be attenuated. The gains $K_i = \begin{pmatrix} K_{1i} & K_{2i} \end{pmatrix}$ and $H_i^T = \begin{pmatrix} H_{1i}^T & H_{2i}^T \end{pmatrix}$ are determined by solving a minimization problem under LMI constraints, given by the following theorem 2.

Theorem 2. *The FTC law (35) minimizing the $\mathcal{L}_2$-gain from $\Delta(t)$ to the tracking and estimation errors $\tilde{e}(t)$ (53) is obtained by finding the symmetric and positive definite matrices $X_1 \in \mathbb{R}^{n \times n}$, $P_2 \in \mathbb{R}^{(n+n_f) \times (n+n_f)}$, matrices $\bar{H}_i \in \mathbb{R}^{(n+n_f) \times p}$, $\bar{K}_{1i} \in \mathbb{R}^{m \times n_f}$ and*

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$K_{2i} \in \mathbb{R}^{m \times n}$ minimizing $\bar{\gamma}$ under the LMI constraints (56), for $i, j = 1, \ldots, r$.

$$
\begin{pmatrix}
\Psi_{ij} & -B_i \overline{K}_j & 0 & \Gamma_{1ij} & 0 & 0 & X_1 \\
* & -2\mu X & \mu I_{n+n_f} & 0 & X & 0 & 0 \\
* & * & Z_i & 0 & 0 & 0 & 0 \\
* & * & * & -\bar{\gamma} I_{2n+n_f} & 0 & \Gamma_{2ij}^T P_2 & 0 \\
* & * & * & * & -I_{n+n_f} & 0 & 0 \\
* & * & * & * & * & -I_{n+n_f} & 0 \\
* & * & * & * & * & * & -I_n
\end{pmatrix} < 0 \quad (56)
$$

with

$$\Psi_{ij} = \mathbb{S}(A_i X_1 - B_i \overline{K}_{1j}) \tag{57}$$

$$Z_i = \mathbb{S}(P_2 \overline{A}_i - \mathcal{H}_i \overline{C}) + I_{n+n_f} \tag{58}$$

$$\overline{K}_j = \begin{pmatrix} \overline{K}_{1j} & K_{2j} \end{pmatrix} \tag{59}$$

$$X = diag(X_1, I_{n_f}) \tag{60}$$

The gains of the observer and of the FTC law, namely H_i and K_j, are obtained by

$$H_i = P_2^{-1} \overline{H}_i, \quad K_{1j} = \overline{K}_{1j} X_1^{-1} \tag{61}$$

the $\mathcal{L}_2$ gain from $\Delta(t)$ to $\tilde{e}$ is given by

$$\gamma = \sqrt{\bar{\gamma}} \tag{62}$$

Proof. The gains $\begin{pmatrix} H_{1i}^T & H_{2i}^T \end{pmatrix}^T$ of the PI observer (36) and $\begin{pmatrix} K_{1i} & K_{2i} \end{pmatrix}$ of the fault tolerant control law (35) are obtained, using the Lyapunov theory, by the minimization of the $\mathcal{L}_2$-gain of the system (53) from $\Delta(t)$ to the tracking and estimation errors $\tilde{e}(t)$. For that purpose, let us chose the following quadratic Lyapunov function

$$V(\tilde{e}(t)) = \tilde{e}^T(t) P \tilde{e}(t), \quad P = P^T > 0 \tag{63}$$

where the symmetric positive definite matrix P is chosen as a block diagonal matrix

$$P = diag(P_1, P_2) \tag{64}$$

where $P_1 \in \mathbb{R}^{n \times n}$ and $P_2 \in \mathbb{R}^{(n+n_f) \times (n+n_f)}$. It is well known [8] that $\tilde{e}(t)$ asymptotically converges toward zero if $\Delta(t) = 0$ and that the $\mathcal{L}_2$-gain from $\Delta(t)$ to the error $\tilde{e}(t)$ is bounded by $\gamma > 0$ if the Lyapunov function $V(\tilde{e})$ satisfies the following inequality

$$\dot{V}(\tilde{e}(t)) + \tilde{e}^T(t)\tilde{e}(t) - \gamma^2 \Delta^T(t)\Delta(t) < 0 \tag{65}$$

From (53), the time derivative of the function $V(\tilde{e}(t))$ is given by

$$\dot{V}(\tilde{e}(t)) = \tilde{e}^T(t) \left(\tilde{A}_{\mu_f \hat{\mu}_f}^T(t) P + P \tilde{A}_{\mu_f \hat{\mu}_f} \right) \tilde{e}(t) + \mathbb{S}\left(\tilde{e}^T(t) P \tilde{\Gamma}_{\mu_f \hat{\mu}_f} \Delta(t) \right) \tag{66}$$

From (54) and (64), the inequality (65) is equivalent to

$$\bar{\xi}^T(t) \mathcal{N}_{\mu_f \hat{\mu}_f}(t) \bar{\xi}(t) < 0 \tag{67}$$

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where

$$\bar{\xi}(t) = \begin{pmatrix} \tilde{e}(t) \\ \Delta(t) \end{pmatrix}, \quad \mathcal{N}_{ij} = \begin{pmatrix} \mathbb{S}(P_1 A_i - P_1 B_i K_{1j}) + I & -P_1 B_i K_j & P_1 \Gamma_{1ij} \\ * & \mathbb{S}(P_2 \bar{A}_i - P_2 H_i \bar{C}) + I & P_2 \Gamma_{2ij} \\ * & * & -\gamma^2 I \end{pmatrix} \tag{68}$$

Since the weighting functions are nonnegative, (65) is satisfied if the matrices $\mathcal{N}_{ij}$ are negative definite. By congruence (lemma 2), for every invertible matrix W, $\mathcal{N}_{ij} < 0$ is equivalent to

$$W^T \mathcal{N}_{ij} W < 0 \tag{69}$$

Defining the matrix $W = diag(P_1^{-1}, X, I_{2n+n_f})$, with $P_1 = X_1^{-1}$, inequality (65) is implied by

$$\begin{pmatrix} \bar{\Psi}_{ij} & -B_i K_j X & \Gamma_{1ij} \\ * & X Z_i X & X P_2 \Gamma_{2ij} \\ * & * & -\gamma^2 I \end{pmatrix} < 0 \tag{70}$$

where

$$\bar{\Psi}_{ij} = \mathbb{S}(A_i X_1 - B_i K_{1j} X_1) + X_1 X_1 \tag{71}$$

$$Z_i = \mathbb{S}(P_2 \bar{A}_i - P_2 H_i \bar{C}) + I \tag{72}$$

If (56) hold, then $Z_i < 0$ and for any scalar μ, the two following inequalities hold.

$$\left(X + \mu Z_i^{-1}\right)^T Z_i \left(X + \mu Z_i^{-1}\right) \leq 0 \Leftrightarrow X Z_i X \leq -\mu(X + X^T) - \mu^2 Z_i^{-1} \tag{73}$$

Using the bounding given by (73) and using a Schur complement on Z_i^{-1}, the non-linear term $X Z_i X$ can be replaced by linear ones. Consequently, the matrix inequality (70) is implied by the following one

$$\begin{pmatrix} \bar{\Psi}_{ij} & -B_i K_j X & 0 & \Gamma_{1ij} \\ * & -2\mu X & \mu I & X P_2 \Gamma_{2ij} \\ * & * & Z_i & 0 \\ * & * & * & -\gamma^2 I \end{pmatrix} < 0 \tag{74}$$

Let us remark that (74) can be written as follows

$$\begin{pmatrix} \bar{\Psi}_{ij} & -B_i K_j X & 0 & \Gamma_{1ij} \\ * & -2\mu X & \mu I & 0 \\ * & * & Z_i & 0 \\ * & * & * & -\gamma^2 I \end{pmatrix} + \mathbb{S}\left(\begin{pmatrix} 0 \\ X \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & P_2 \Gamma_{2ij} \end{pmatrix} \right) < 0 \tag{75}$$

From the lemma 1, (75) is implied by

$$\begin{pmatrix} \bar{\Psi}_{ij} & -B_i K_j X & 0 & \Gamma_{1ij} \\ * & -2\mu X + X X^T & \mu I & 0 \\ * & * & Z_i & 0 \\ * & * & * & -\gamma^2 I + \Gamma_{2ij}^T P_2 P_2 \Gamma_{2ij} \end{pmatrix} < 0 \tag{76}$$

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and by using the Schur complement, we obtain

$$
\begin{pmatrix}
\overline{\Psi}_{ij} & -B_iK_jX & 0 & \Gamma_{1ij} & 0 & 0 \\
* & -2\mu X & \mu I & 0 & X & 0 \\
* & * & Z_i & 0 & 0 & 0 \\
* & * & * & -\gamma^2 I & 0 & \Gamma_{2ij}^T P_2 \\
* & * & * & * & -I & 0 \\
* & * & * & * & * & -I
\end{pmatrix} < 0, \ i,j = 1,...,r \qquad (77)
$$

Defining $X = diag(P_1^{-1}, I_{n_f})$ and with the following variable changes

$$
\overline{H}_i = P_2 H_i, \ \ \overline{K}_{1j} = K_{1j}X_1, \ \ \overline{\gamma} = \gamma^2 \qquad (78)
$$

and a Schur complement on the term $X_1 X_1$ in $\overline{\Psi}_{ij}$, the LMIs in theorem 2 are obtained. As a consequence (56) implies (65) and then ensures that the $\mathcal{L}_2$-gain from $\Delta(t)$ to $\tilde{e}(t)$ is bounded by γ. □

5.2 Time varying faults

The assumption that the fault signal is constant over the time is restrictive, but in many practical situations where the faults are slowly time-varying signals, the estimation of the faults is correct, and the previous proposed FTC scheme can be applied. In the case where the faults are not slowly time-varying or constant, the Proportional Integral Observer (PIO) can be replaced by a Proportional Multiple Integral Observer (PMIO). Such an observer is able to estimate a large class of time-varying signals satisfying the following assumption

$$
f^{(q+1)} = 0 \qquad (79)
$$

The principle of this observer is based on the estimation of all the first q^{th} derivatives of the signal $f(t)$. This observer can also be extended to the case where $f^{(q+1)}$ is bounded.

Let consider the system (34) with a fault in the general polynomial form

$$
f(t) = a_0 + a_1 t + a_2 t^2 + ... + a_q t^q \qquad (80)
$$

Let consider $d_0(t) = \dot{f}(t), \ \ d_1(t) = \ddot{f}(t),..., \ \ d_{q-1}(t) = f^{(q)}(t)$. Thus, the system can be transformed into an augmented form

$$
\begin{cases}
\dot{\tilde{x}}_a(t) = \sum\limits_{i=1}^{r} \mu_i(x_f(t)) \left(\tilde{A}_i \tilde{x}_a(t) + \tilde{B}_i u_f(t) \right) \\
y(t) = \tilde{C}\tilde{x}_a(t)
\end{cases} \qquad (81)
$$

where

$$
\tilde{x}_a(t) = \begin{pmatrix} x_f(t) \\ d_0(t) \\ \vdots \\ d_{q-2}(t) \\ d_{q-1}(t) \end{pmatrix}, \ \tilde{A}_i = \begin{pmatrix} A_i & G_i & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & I \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \ \tilde{B}_i = \begin{pmatrix} B_i \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \tilde{C} = \begin{pmatrix} C & R & 0 & ... & 0 \end{pmatrix}
$$

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The observer simultaneously estimating the state $x_f(t)$, the faults $f(t)$ and the successive derivatives is given in the following form

$$\begin{cases} \dot{\hat{x}}_a(t) = \sum_{i=1}^{r} \mu_i(\hat{x}_f(t)) \left(\tilde{A}_i \hat{x}_a(t) + \tilde{B}_i u_f(t) + \tilde{H}_i(y(t) - \hat{y}(t)) \right) \\ \hat{y}(t) = \tilde{C} \hat{x}_a(t) \end{cases} \tag{82}$$

Using (35), (81) and (82), the tracking error $e(t) = x(t) - x_f(t)$ and the augmented state estimation error $e_a(t) = \tilde{x}_a(t) - \hat{x}_a(t)$ are given by

$$\begin{pmatrix} \dot{e}(t) \\ \dot{e}_a(t) \end{pmatrix} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_f(t)) \mu_j(\hat{x}_f(t)) \tilde{\tilde{A}}_{ij} \begin{pmatrix} e(t) \\ e_a(t) \end{pmatrix} + \tilde{\tilde{\Gamma}}_{ij} \Delta(t) \tag{83}$$

where

$$\tilde{\tilde{A}}_{ij} = \begin{pmatrix} A_i - B_i K_{1j} & -B_i K_j \\ 0 & \tilde{A}_i - \tilde{H}_i \tilde{C} \end{pmatrix}, \quad \tilde{\tilde{\Gamma}}_{ij} = \begin{pmatrix} \Gamma_{1ij} \\ \tilde{\Gamma}_{2ij} \end{pmatrix}, \quad \tilde{\Gamma}_{2ij} = \begin{pmatrix} \Gamma_{2ij} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus, the structure of the state equations is the same as those expressed in the case of constant faults. The synthesis of the gains of the controller and those of the observer are obtained by solving the LMIs given in the theorem 2.

6 Application to the fault tolerant control of vehicle lateral dynamics

Security and assistance in vehicles are important especially in dangerous situations like cornering and land keeping with high speed. For that purpose, this example deals with the problem of fault tolerant control of lateral dynamics of a vehicle in cornering situation.

6.1 System modeling

Let us consider the nonlinear model of the vehicle lateral dynamics described in the figure 3 and given by the differential equations (84).

$$\begin{cases} m(\dot{v}_y(t) + v_x r(t)) = 2 (F_f(t) + F_r(t)) \\ J\dot{r}(t) = 2 (a_f F_f(t) - a_r F_r(t)) \end{cases} \tag{84}$$

where $v_x(t)$ and $v_y(t)$ are the longitudinal and lateral velocities ($v_y(t) = \beta v_x(t)$), $r(t)$ is the yaw rate, β denotes the side slip angle, m and J are the mass and the yaw moment of inertia respectively, a_f and a_r are the distances of the front and rear axle from the center of gravity. $F_f(t)$ and $F_r(t)$ are front and rear lateral forces. Different kinds of tire forces models are proposed in the literature, we can cite the Dugoff's model or Pacejka's model [40] which are empirical models obtained from real datas. The last
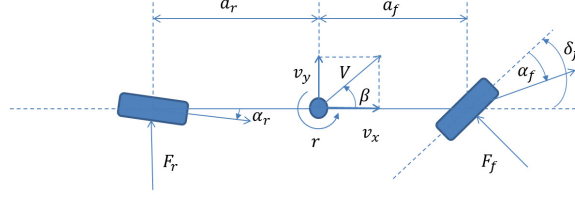
Figure 3: Single track model of the vehicle

one is more used in vehicle and tire construction. For our study, the forces are modeled with the magic formula as given in [40] with the following equations

$$\begin{cases} F_f(t) = D_f \sin\left(C_f \tan^{-1}\left(B_f(1-E_f)\alpha_f(t) + E_f \tan^{-1}(B_f\alpha_f(t))\right)\right) \\ F_r(t) = D_r \sin\left(C_r \tan^{-1}\left(B_r(1-E_r)\alpha_r(t) + E_r \tan^{-1}(B_r\alpha_r(t))\right)\right) \end{cases} \quad (85)$$

The coefficient D_i, C_i, B_i and E_i $(i = f, r)$ depend on the characteristics of the tire and the road adhesion. The parameters α_f and α_r represent tire slip angles at the front and rear of the vehicle. These quantities depend on the state of the system and are expressed by

$$\begin{cases} \alpha_f(t) = -\frac{v_y(t)}{v_x(t)} - \tan^{-1}\left(\frac{a_f}{v_x(t)}r(t)\cos\left(\frac{v_y(t)}{v_x(t)}\right)\right) + \delta_f(t) \\ \alpha_r(t) = -\frac{v_y(t)}{v_x(t)} - \tan^{-1}\left(\frac{a_r}{v_x(t)}r(t)\cos\left(\frac{v_y(t)}{v_x(t)}\right)\right) \end{cases} \quad (86)$$

where $\delta_f(t)$ is the front steering angle. A T-S simplified model as given in [39] is represented by

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i(\alpha_f(t)) \left(A_i x(t) + B_i u(t)\right) \\ y(t) = Cx(t) \end{cases} \quad (87)$$

where $x^T(t) = [v_y(t) \ r(t)]$, $u(t) = \delta_f(t)$. The output $y(t)$ contains the yaw rate given by the inertial unit. The matrices defining the model are

$$A_1 = \begin{pmatrix} -6.9426 & -0.8775 \\ 26.5175 & -7.8343 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 3.4892 \\ 39.2914 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -0.4735 & -0.9971 \\ 0.6356 & -0.4921 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.2767 \\ 3.1155 \end{pmatrix}$$

$$C = (0 \ 1)$$

The weighting function are given by the following equations

$$\mu_i(\alpha_f(t)) = \frac{\beta_i(\alpha_f(t))}{\sum_{i=1}^{2} \beta_i(\alpha_f(t))}, i = 1, 2 \quad (88)$$

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with

$$\beta_i(\alpha_f(t)) = \frac{1}{\left(1 + \left|\frac{|\alpha_f(t)| - c_i}{a_i}\right|\right)^{2b_i}} \tag{89}$$

The parameters a_i, b_i and c_i are the same as those used in [39], which are obtained by Levenberg Marquard optimization.

$$a_1 = 0.5077 \qquad\qquad a_2 = 0.4748$$
$$b_1 = 3.1893 \qquad\qquad b_2 = 5.3907$$
$$c_1 = -0.4356 \qquad\qquad c_2 = 0.5622$$

The weighting function μ_i depend on the slip angle α_f which depend on state variables that are not completely measured (as presented in (86)). Finaly, the fault free model of the vehicle lateral dynamics is given by the following equations

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{2} \mu_i(x(t)) \left(A_i x(t) + B_i u(t)\right) \\ y(t) = Cx(t) \end{cases} \tag{90}$$

6.2 Fault tolerant control design for vehicle lateral dynamics

Let us consider an additive actuator fault signal affecting the vehicle in the form

$$f(t) = \begin{cases} 0, & if\ t < 10 \\ 0.1\sin(0.314t), & if\ 10 \le t \le 14.95 \\ 0.1, & if\ t > 14.95 \end{cases} \tag{91}$$

The faulty system is then expressed as follows

$$\begin{cases} \dot{x}_f(t) = \sum\limits_{i=1}^{2} \mu_i(x_f(t)) \left(A_i x_f(t) + B_i(u_f(t) + f(t))\right) \\ y_f(t) = Cx_f(t) \end{cases} \tag{92}$$

One can note that this system corresponds to (34) with $G_i = B_i$ and $R = 0$. The control observer based control law given by the equations (35) and (36) is designed by solving the LMI optimization problem defined in the theroem 2. Simulations are realized by considering a random noise with maximal magnitude 0.01 as an output measurements. Figure 4 depicts the vehicle state estimation. The fault, its estimate and the input with and without FTC are displayed on figure 5. The fault is correctly estimated. The control input is clearly reconfigurated according to the fault estimation, as shown by the deviation between the FTC input and the nominal control input. Figure 6 illustrates a comparison between the states of the reference model (without fault), the states of the faulty vehicle without FTC and finally the states of the vehicle with FTC. Clearly, the proposed strategy is robust with respect to actuator additive fault $f(t)$ since the state trajectories of the reference model and those of the system with FTC are quite indistinguishable. Moreover the fault distinctly acts on the system trajectory since the system states without FTC significantly deviate from the reference ones. Moreover, the FTC scheme provides good results in the presence of measurement noises as shown in the figures.
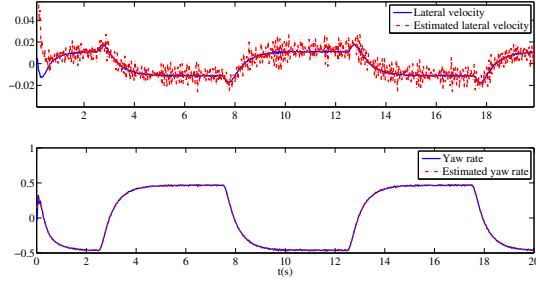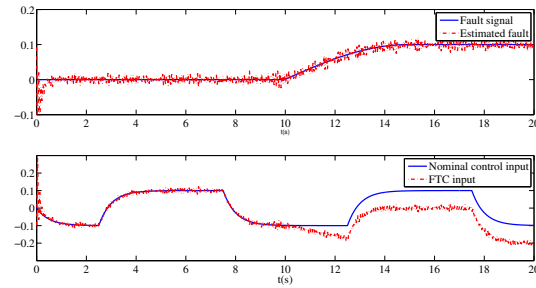
Figure 4: State estimation

Figure 5: Fault estimation and fault tolerant control signal

7 Conclusion

This chapter is dedicated to the design of new approaches for nonlinear observer and fault tolerant controller designs. The considered systems are modeled with a T-S structure with unmeasurable premise variables. Observer design is made by using the mean value theorem (MVT) in order to re-write the state estimation error in form of autonomous system. After the Lyapunov theory is used to analyze the stability of the state estimation error dynamics and then to derive LMI conditions. The second problem which is studied is the fault tolerant control design. The strategy is based on the use of a reference model which is given by the fault-free system model. The proposed control law is then designed to minimize the deviation of the system state compared to the reference state, even in the presence of fault(s). This control law uses the nominal control input developed for the system in fault-free case and two additional terms. The first term is related to the estimated fault and the second one corresponds to the trajectory tracking error. The stability of the tracking error model is studied with the Lyapunov theory and a quadratic function that allows to derive conditions ensuring the convergence of the state and fault estimation errors and trajectory tracking error toward zero. The existence conditions are expressed in terms of LMI that can be solved with classical dedicated softwares. Two examples are proposed both for state estimation and fault tolerant control. The first one concerns the state estimation for a Lorenz system
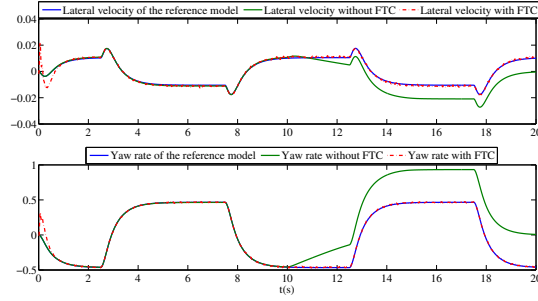
Figure 6: States of the vehicle : reference states, states without FTC and states with FTC

and the second one concerns the fault tolerant control of vehicle lateral dynamics.

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