Observers design for uncertain Takagi-Sugeno system with unmeasurable premise variables and unknown input. Application to a wastewater treatment plant

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Abstract

This article aims the observer synthesis for uncertain nonlinear systems and affected by unknown inputs, represented under the multiple model (MM) formulation with unmeasurable premise variables. A proportional integral observer (PIO) is considered. In order to design such an observer, the nonlinear system is transformed into an equivalent MM form. The Lyapunov method, expressed through linear matrix inequality (LMI) formulation, is used to describe the stability analysis and for observer synthesis. An application to a model of Wastewater Treatment Plant (WWTP) is considered and the performances of the proposed approach are illustrated through numerical results.
Keywords: nonlinear system, model uncertainties, state estimation, unknown input estimation, unmeasurable premise variables, LMI, WWTP

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1. Introduction

In the field of the observer/controller synthesis, the extension of linear methods to nonlinear systems is generally a difficult problem. Several approaches are reported in the literature along the last decades in order to design observers for nonlinear systems: the extended Kalman filtering [1, 2], the method based on Lyapunov functions [3, 4], methods based on coordinate transformation using canonical observer forms [5, 6], high gain observers [7, 8] or sliding mode observers [9, 10] are only a few of them.

The multiple model (MM) [11] -also called in the literature Takagi-Sugeno fuzzy model [12]- has received a special attention in the last two decades, in order to overcome the difficulty already mentioned. The MM is mainly based on the idea of a complexity reduction of nonlinear systems, by aggregating linear submodels using weighting functions [12]. Several techniques [13, 14, 15, 16] were developed in order to obtain such a structure from a general representation of a nonlinear system. Then the MM approach is a mean to deal with nonlinear systems and to design observer for such systems [17, 18, 19, 20, 21, 22]. In this paper, the MM formulation is obtained by applying a method proposed in [23] to represent nonlinear systems into an equivalent MM. Only the general steps of this technique are reminded in this paper. The major inconveniences of the previous works are avoided: the transformation is realized without loss of information, the obtained system has exactly the same state trajectory as the initial system, the choice of different linearization points is no longer necessary.

The MM under study in this paper involves unmeasurable premise variables depending on the state variables. Most of the existing works, dedicated to MM in general and to observer design based on MM in particular [16, 22, 18, 24], are established for MM with measurable premise variables (inputs/outputs). But, in many practical situations, these premise variables depend on the state variables, that are not always accessible. Recently, a few works [17, 21, 25] are devoted to the case of unmeasurable premise variables.

A proportional integral observer approach for uncertain nonlinear systems with unknown inputs presented under a MM form with unmeasurable premise variables is proposed in this paper. The state and unknown input estimation given by this observer is made simultaneously and the influence of the model uncertainties is minimized through a $\mathcal{L}_2$ gain. The convergence conditions of the state and unknown input estimation errors are expressed through LMIs (Linear Matrix Inequalities) by using the Lyapunov method and the $\mathcal{L}_2$ approach. An extension of this approach is finally discussed and it concerns the uncertain nonlinear systems also affected by some unknown inputs which estimates are not needed. In this case, the previous result can readily be adapted in order to estimate both state and a part of unknown inputs (for which an estimation is needed) while minimizing the influence of not needed unknown inputs on the estimation errors.

The chosen application is a wastewater treatment plant (WWTP). The strong nonlinearity of a WWTP is due to the variations of the wastewater flow rate and composition, to reactions that vary in time in a mixed culture of microorganisms. In order to model this highly complex bio-chemical process, several models were proposed in the literature: ASM1 (Activated Sludge Model No.1) [26, 27], ASM2 [28], ASM3 [29]. In order to deal with the complexity of such models, different reduced models for the activated sludge plant were proposed along the last decades [30, 31, 32, 33]. In this article, a nonlinear reduced model with six states inspired by [34] is chosen. Recently, observer design for the activated sludge wastewater treatment process was proposed in [35], where a reduced model was also considered. In [35], the observer design is done for a class of nonlinear Lipschitz discrete-time systems using the linear parameter varying approach. Thus, this approach needs the respect of the Lipschitz property of the nonlinear part, that may constitute a quite conservative property. Our approach avoids this assumption. In [35], the linear parameter varying term depends only on an online accessible scheduling variable. The approach proposed in this article is placed in a more general situation, that was discussed previously: unmeasurable premise variables are considered in the MM form. The Cost Benchmark was considered for simulation results. This benchmark has been proposed by the European program Cost 624 for the evaluation of control strategies in wastewater treatment plants [36].

The paper is organized as follows: in section 2 are introduced the main tools used in this paper to model nonlinear systems: the MM form of nonlinear systems. Section 3 is devoted to the proportional integral observer design. In section 4 an application to a model of wastewater treatment plants (ASM1) is proposed and the numerical results illustrate the performances of the proposed observer approach. Finally, the paper ends with some conclusions and future works, in section 5.

2. Multiple model representation of nonlinear systems

Generally, a dynamic nonlinear system can be described by the following ordinary differential equations:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned} \tag{1}$$

The MM approach allows to represent any nonlinear dynamic system in a compact set of the state space with a convex combination of linear submodels:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x,u) \left[A_i x(t) + B_i u(t)\right] \\ y(t) &= \sum_{i=1}^{r} \mu_i(x,u) \left[C_i x(t) + D_i u(t)\right] \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^l$ the output vector, A_i, B_i, C_i and D_i are constant matrices of appropriate dimensions. The functions $\mu_i(x,u)$ also called the activating functions represent the weights of the submodels $\{A_i, B_i, C_i, D_i\}$ in the global model and they have the following properties:

$$\sum_{i=1}^{r} \mu_i(x,u) \;=\; 1 \quad \text{and} \quad \mu_i(x,u) \;\geq\; 0, \;\; \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \tag{3}$$

In order to deduce the MM form (2), a method giving an equivalent rewriting of the nonlinear system (1) is used [23]. The main steps of the rewriting procedure are given in the following, further details can be found in [23].

Firstly, the system (1) is transformed into a quasi-Linear Parameter Varying (quasi-LPV) form:

$$\begin{aligned} \dot{x}(t) &= A(x(t), u(t))\, x(t) + B(x(t), u(t))\, u(t) \\ y(t) &= C(x(t), u(t))\, x(t) + D(x(t), u(t))\, u(t) \end{aligned} \tag{4}$$

Secondly, the nonlinear entries of the matrices A, B, C and/or D are considered as "premise variables" and denoted $z_j(x, u)(j = 1, ..., q)$. Several choices of these premise variables are possible due to the existence of different equivalent quasi-LPV forms (for details on the selection procedure see [23]).
Thirdly, a convex polytopic transformation is performed for all the premise variables $(j = 1, ..., q)$, as follows:

$$z_j(x, u) = F_{j,1}(z_j(x, u))\, z_{j,1} + F_{j,2}(z_j(x, u))\, z_{j,2} \tag{5}$$

where the scalars $z_{j,1}$ and $z_{j,2}$ are defined by

$$\begin{aligned} z_{j,1} &= \max_{x,u} \{z_j(x, u)\} \\ z_{j,2} &= \min_{x,u} \{z_j(x, u)\} \end{aligned} \tag{6}$$

and where the partition functions involved in equation (5) are defined by

$$F_{j,1}(z_j(x, u)) = \frac{z_j(x, u) - z_{j,2}}{z_{j,1} - z_{j,2}} \tag{7a}$$

$$F_{j,2}(z_j(x, u)) = \frac{z_{j,1} - z_j(x, u)}{z_{j,1} - z_{j,2}} \tag{7b}$$

For q premise variables, $r = 2^q$ submodels will be obtained. By multiplying the functions F_{j,σ_i^j} the weighting functions are obtained:

$$\mu_i(x, u) = \prod_{j=1}^{q} F_{j,\sigma_i^j}(z_j(x, u)) \tag{8}$$

Considering definition (7), the reader should remark that these functions respect the conditions (3).
The indexes σ_i^j $(i = 1, ..., 2^q$ and $j = 1, ..., q)$ are equal to 1 or 2 and indicates which partition of the j^{th} premise variable ($F_{j,1}$ or $F_{j,2}$) is involved in the i^{th} submodel.

The constant matrices A_i, B_i, C_i and D_i ($i = 1, ..., 2^q$) are obtained by replacing the premise variables z_j in the matrices A, B, C and D, with the scalars defined in (6):

$$
\begin{aligned}
A_i &= A(z_{1,\sigma_i^1}, ..., z_{q,\sigma_i^q}) && \text{(9a)}\\
B_i &= B(z_{1,\sigma_i^1}, ..., z_{q,\sigma_i^q}) && \text{(9b)}\\
C_i &= C(z_{1,\sigma_i^1}, ..., z_{q,\sigma_i^q}) && \text{(9c)}\\
D_i &= D(z_{1,\sigma_i^1}, ..., z_{q,\sigma_i^q}) && \text{(9d)}
\end{aligned}
$$

3. Observer design

This section is devoted to the state estimation of the multiple model (2). In fact a more general situation will be analyzed since unknown input and model uncertainties are envisaged. The following uncertain Takagi-Sugeno system affected by unknown inputs is considered

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x,u) \left((A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))u(t) + E_i d(t) \right) && \text{(10a)}\\
y(t) &= Cx(t) + Gd(t) && \text{(10b)}
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^{n_u}$ is the known input, $d(t) \in \mathbb{R}^{n_d}$ is the unknown input, $y(t) \in \mathbb{R}^{n_y}$ is the measured output and the matrices of appropriate dimensions are known and constant excepted $\Delta A_i(t)$ and $\Delta B_i(t)$ that satisfy the following equations

$$
\begin{aligned}
\Delta A_i(t) &= M_i^a F_a(t) N_i^a, \quad \text{with} \quad F_a^T(t) F_a(t) \leq I && \text{(11a)}\\
\Delta B_i(t) &= M_i^b F_b(t) N_i^b, \quad \text{with} \quad F_b^T(t) F_b(t) \leq I && \text{(11b)}
\end{aligned}
$$

where both $F_a(t) \in \mathbb{R}^{f_1 \times f_1}$ and $F_b(t) \in \mathbb{R}^{f_2 \times f_2}$ are unknown and time varying. One can note that the activating functions depend on the system state that is not available to the measurement. In the sequel, the following assumption is made

Assumption 1. *The unknown input is constant :*

$$\dot{d}(t) = 0 \tag{12}$$

It is well known in proportional integral observer (PIO) design that, although this assumption is needed for the theoretical proof of the estimation error convergence, it can be relaxed in practical applications [37]. For instance, one will see, in section 4, that good estimation results are obtained even with time varying unknown input.

In order to estimate both the system state and the unknown input, the following PIO is proposed:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t), u(t)) \left(A_i \hat{x}(t) + B_i u(t) + E_i \hat{d}(t) + L_i^P (y(t) - \hat{y}(t)) \right) \tag{13a}$$

$$\dot{\hat{d}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t), u(t)) L_i^I (y(t) - \hat{y}(t)) \tag{13b}$$

$$\hat{y}(t) = C\hat{x}(t) + G\hat{d}(t) \tag{13c}$$

The observer design reduces to finding the gains L_i^P and L_i^I such that the state and unknown input estimation error obey to a stable generating system.

Notation 1. *The symbol $*$ in a block matrix denotes the blocks induced by symmetry. For any square matrix M, $\mathbb{S}(M)$ is defined by $\mathbb{S}(M) = M + M^T$.*

Theorem 1. *The observer* (13) *estimating the state and unknown input of the system* (10) *and minimizing the* $\mathcal{L}_2$*-gain of the known and unknown inputs on the state and unknown input estimation error is obtained by finding symmetric positive definite matrices* $P_1 \in \mathbb{R}^{(n+n_d)\times(n+n_d)}$ *and* $P_2 \in \mathbb{R}^{n\times n}$*, matrices* $\overline{P}_j \in \mathbb{R}^{(n+n_d)\times n_y}$ *and positive scalars* ε_{1i} *and* ε_{2i} *that minimize the scalar* $\overline{\gamma}$ *under the following LMI constraints*

$$\mathcal{M}_{ij} < 0,\ i,j = 1,\ldots,r \tag{14}$$

where $\mathcal{M}_{ij}$ *is defined by*

$$\mathcal{M}_{ij} = \begin{bmatrix} \Theta_{ij}^{11} & \Theta_{ij}^{12} & 0 & \Theta_{ij}^{14} & P_1\overline{M}_i^a & P_1\overline{M}_i^b \\ * & \Theta_{ij}^{22} & P_2B_i & P_2E_i & P_2M_i^a & P_2M_i^b \\ * & * & \Theta_{ij}^{33} & 0 & 0 & 0 \\ * & * & * & -\overline{\gamma}I_{n_d} & 0 & 0 \\ * & * & * & * & -\varepsilon_{1i}I_{f_1} & 0 \\ * & * & * & * & * & -\varepsilon_{2i}I_{f_2} \end{bmatrix} \tag{15}$$

with

$$\begin{aligned} \Theta_{ij}^{11} &= I_{n+n_d} + \mathbb{S}(P_1\overline{A}_j - \overline{P}_j\overline{C}) & \Theta_{ij}^{12} &= P_1(\tilde{A}_i - \tilde{A}_j) & \Theta_{ij}^{14} &= P_1(\tilde{E}_i - \tilde{E}_j) \\ \Theta_{ij}^{22} &= \varepsilon_{1i}N_i^{aT}N_i^a + \mathbb{S}(P_2A_i) & \Theta_{ij}^{33} &= \varepsilon_{2i}N_i^{bT}N_i^b - \overline{\gamma}I_{n_u} \end{aligned}$$

and where the overlined and tilded matrices are defined by

$$\begin{aligned} \overline{C} &= \begin{bmatrix} C & G \end{bmatrix}, & \overline{A}_i &= \begin{bmatrix} A_i & E_i \\ 0 & 0 \end{bmatrix}, & \tilde{A}_i &= \begin{bmatrix} A_i \\ 0 \end{bmatrix}, \\ \tilde{E}_i &= \begin{bmatrix} E_i \\ 0 \end{bmatrix}, & \overline{M}_i^a &= \begin{bmatrix} M_i^a(t) \\ 0 \end{bmatrix}, & \overline{M}_i^b &= \begin{bmatrix} M_i^b(t) \\ 0 \end{bmatrix} \end{aligned}$$

The observer gains are obtained by:

$$L_j = \begin{bmatrix} L_j^P \\ L_j^I \end{bmatrix} = P_1^{-1}\overline{P}_j$$

Proof. See proof in Appendix A. □

Remark 1. If the system is also affected by unknown input which estimates are not needed, the system (10) becomes

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t)) \left((A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))u(t) + E_i d(t) + F_i w(t) \right) \tag{16a}$$

$$y(t) = Cx(t) + Gd(t) + Hw(t) \tag{16b}$$

where $w(t) \in \mathbb{R}^{n_w}$ denotes these unknown inputs. In this case, the previous result can readily be adapted in order to estimate both $x(t)$ and $d(t)$ while minimizing the influence of $w(t)$ on the estimation errors. One should easily see that the matrices $\mathcal{M}_{ij}$ in (14) should be replaced by $\tilde{\mathcal{M}}_{ij}$ defined by

$$\tilde{\mathcal{M}}_{ij} = \begin{bmatrix} \mathcal{M}_{ij} & \Psi_{ij} \\ * & -\overline{\gamma} I_{n_w} \end{bmatrix} \tag{17}$$

with $\Psi_{ij}^T = \left[(\overline{P}_j \overline{H} - P_1 \overline{F}_i)^T \quad F_i^T P_2 \quad 0 \quad 0 \right]$, $\overline{H}^T = \left[H^T \ 0 \right]$ and $\overline{F}_i^T = \left[F_i^T \ 0 \right]$.

4. Application to a wastewater treatment plant model

In this section the proposed observer approach is applied to a model of a wastewater treatment plant -the ASM1 model [26, 31]- in order to reconstruct the state variables and the unknown inputs. First the wastewater treatment process and the model used are presented.

4.1. Process description and ASM1 model

The activated sludge wastewater treatment is widely used and studied in the last two decades [26, 27, 30, 31, 33, 34, 35, 36]. It consists in mixing waste water with a rich mixture of bacteria in order to degrade the pollutants contained in the water.

The polluted water circulates in the basin of aeration in which the bacterial biomass degrades the polluted matter. Micro-organisms gather together in colonial structures called flocs and produce sludges. The mixed liqueur is then sent to the clarifier where the separation of the purified water and the flocs is made by gravity. A fraction of settled sludges is recycled towards the bioreactor to maintain its capacity of purification. The purified water is thrown back in the natural environment.

In this work, we consider only a part of the Cost Benchmark. The Cost Benchmark has been proposed by the European program Cost 624 for the evaluation of control strategies in wastewater treatment plants [36]. The Benchmark is based on the most common wastewater treatment plant: a continuous flow activated sludge plant, performing nitrification and pre-nitrification. Usually, a configuration with a single tank with a settler/clarifier was developed.

The data used for simulation are generated with the complete ASM1 model ($n = 13$) [26, 33], in order to represent a realistic behavior of a WWTP. The observer design is based on a reduced model ($n = 6$) [34], in order to ease the obtaining of the MM representation and to ensure a solution for the associated LMIs. The estimation results represent a comparison between the signals of the complete ASM1 model and those of the observer. Even if the observer design is based on a MM form of the reduced ASM1 model ($n = 6$) it will be seen that the estimation results are satisfactory.

The functioning principle of the process is briefly described after. The simplified diagram, given in figure 1, includes a tank of aeration (bioreactor) and a clarifier. In this figure q_{in} represents the fresh water input flow, q_{out} the bioreactor output flow, q_a the air flow, q_R and q_W are respectively the clarifier recycled and the rejected flow. The reactor volume V is assumed to be constant and thus:

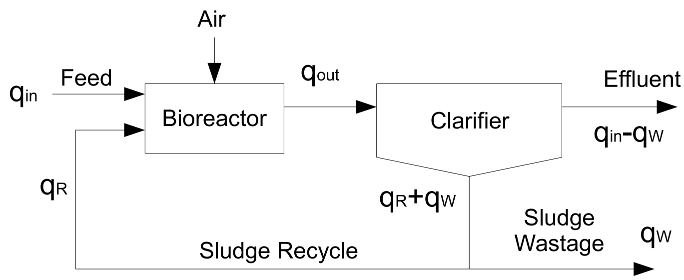

Figure 1: Wastewater treatment process diagram

$$q_{out}(t) \quad = \quad q_{in}(t) + q_R(t) \qquad (18)$$

In general, $q_R(t)$ and $q_W(t)$ represent fractions of the input flow $q_{in}(t)$:

$$q_R(t) \quad = \quad f_R\, q_{in}(t), \qquad 1 \leq f_R \leq 2 \qquad (19)$$

$$q_W(t) \quad = \quad f_W\, q_{in}(t), \qquad 0 < f_W < 1 \qquad (20)$$

In the bioreactor, under the homogeneity hypothesis, the equations of mass conservation for the various constituents, involving the reaction and the input/output balance parts, are given by:

$$\dot{x}(t) \quad = \quad r(x(t)) + \frac{q_{in}(t)x_{in}(t) + q_R(t)x_R(t) - [q_{in}(t) + q_R(t)]x(t)}{V} \qquad (21)$$

where $x = [S, X]^T$ is the vector of soluble (S) and particular (X) constituents, $x_R = [S_R, X_R]^T$ is the vector of constituents corresponding to recycled sludges. The reactor homogeneity hypothesis are often given as [31]:

$$x_{out}(t) = x(t) \tag{22}$$

The clarifier/settler is assumed to be perfect i.e. no sludge leaves the clarifier tank by the overflow. In this case it can be written:

$$S_R(t) = S(t) \tag{23}$$
$$X_R(t) = \frac{q_{in}(t) + q_R(t)}{q_R(t) + q_W(t)} X(t) \tag{24}$$

The reaction part of (21) can be put under the form:

$$r(x(t)) = C_{coef}\ \Phi(x(t)) \tag{25}$$

where the matrix C_{coef} of stoichiometric coefficients and the vector $\Phi(x(t))$ of process kinetics are explicitly defined in [31].

Some simplifying assumptions are applied for ASM1 reduction: simplification with respect to components is considered [34]. For the sake of brevity, we only consider the biological removal of carbon and nitrogen from wastewater. Thus, the simplified model involve the following six components: soluble carbon S_S, particulate X_S, oxygen S_O, heterotrophic biomass X_{BH}, ammonia S_{NH}, nitrate S_{NO} and autotrophic biomass X_{BA}. The following components are not considered: inert components (S_I, X_I, X_P, the alkalinity (S_{alk}). The dynamic of the suspended organic nitrogen (X_{ND}) and the ammonia production from organic nitrogen (S_{ND}) is neglected.

Since it is not practically possible to distinguish the soluble part S_S and the particulate part X_S from the measurement of the chemical oxygen demand (COD), a single organic compound (denoted X_{DCO}) will be considered by adding the two compound concentrations [34].

The following state vector is thus considered:

$$x = [X_{DCO}, S_O, S_{NH}, S_{NO}, X_{BH}, X_{BA}]^T \tag{26}$$

The reduced matrix $C_{coef} \in \mathbb{R}^{6\times 5}$ in (25) is defined by:

$$C_{coef} = \begin{bmatrix} -\frac{1}{Y_H} & -\frac{1}{Y_H} & 0 & 1-f_P & 1-f_P \\ \frac{Y_H-1}{Y_H} & 0 & \frac{Y_A-4.57}{Y_A} & 0 & 0 \\ -i_{XB} & -i_{XB} & -i_{XB}-\frac{1}{Y_A} & i_{XB}-f_P\, i_{XP} & i_{XB}-f_P\, i_{XP} \\ 0 & \frac{Y_H-1}{2.86\, Y_H} & \frac{1}{Y_A} & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \tag{27}$$

and the vector $\Phi(x(t)) = [\varphi_1(t), \cdots, \varphi_5(t)]^T \in \mathbb{R}^5$ is given by:

$$\begin{aligned} \varphi_1(t) &= \mu_H \frac{X_{DCO}(t)}{K_{DCO}+X_{DCO}(t)} \frac{S_O(t)}{K_{OH}+S_O(t)} X_{BH}(t) \\ \varphi_2(t) &= \mu_H \eta_{NOg} \frac{X_{DCO}(t)}{K_{DCO}+X_{DCO}(t)} \frac{S_{NO}(t)}{K_{NO}+S_{NO}(t)} \frac{K_{OH}}{K_{OH}+S_O(t)} X_{BH}(t) \\ \varphi_3(t) &= \mu_A \frac{S_{NH}(t)}{K_{NH,A}+S_{NH}(t)} \frac{S_O(t)}{K_{O,A}+S_O(t)} X_{BA}(t) \\ \varphi_4(t) &= b_H X_{BH}(t) \\ \varphi_5(t) &= b_A X_{BA}(t) \end{aligned} \tag{28}$$

where

$$K_{DCO} = K_S \frac{X_{DCO}}{S_S} = \frac{K_S}{f_{SS}}$$

Hypothesis 1. *It is supposed that the dissolved oxygen concentration at the reactor input ($S_{O,in}$) is null. It is also supposed that $S_{NO,in} \cong 0$ and*

$X_{BA,in} \cong 0$, which is in conformity with the benchmark of European Program Cost 624 [36]. In practice and in particular in the Bleesbruck station from Luxembourg, the concentrations $X_{DCO,in}$, $X_{BH,in}$ and $S_{NH,in}$ are not measured online. Thus, a frequently used approximation is to replace these concentrations with their respective daily mean values. A daily mean value will be considered for $X_{DCO,in}$ -known input- and the concentrations $X_{BH,in}$ and $S_{NH,in}$ will be considered as unknown inputs. The measurements of the four concentrations in the reactor output (X_{DCO}, S_O, S_{NH} and S_{NO}) are available online.

Remark 2. Another option exists that is to chose the daily mean value for $X_{BH,in}$ and consider $X_{DCO,in}$ as unknown input, since the estimation approach is based on the idea of minimizing the $\mathcal{L}_2$-gain of the known and unknown inputs on the state and unknown input estimation error.

Let us consider the explicit form of the ASM1 model (21) characterized by the reduced state vector (26) and the stoichiometric matrix (27) as follows:

$$\begin{aligned}
\dot{X}_{DCO}(t) &= -\frac{1}{Y_H}[\varphi_1(t)+\varphi_2(t)] + (1-f_P)(\varphi_4(t)+\varphi_5(t)) + D_1(t)\\
\dot{S}_O(t) &= \frac{Y_H-1}{Y_H}\varphi_1(t) + \frac{Y_A-4.57}{Y_A}\varphi_3(t) + D_2(t)\\
\dot{S}_{NH}(t) &= -i_{XB}[\varphi_1(t)+\varphi_2(t)] - \left(i_{XB}+\frac{1}{Y_A}\right)\varphi_3(t)\\
&\quad +(i_{XB}-f_P\, i_{XP})[\varphi_4(t)+\varphi_5(t)] + D_3(t) \qquad (29)\\
\dot{S}_{NO}(t) &= \frac{Y_H-1}{2.86Y_H}\varphi_2(t) + \frac{1}{Y_A}\varphi_3(t) + D_4(t)\\
\dot{X}_{BH}(t) &= \varphi_1(t)+\varphi_2(t)-\varphi_4(t)+D_5(t)\\
\dot{X}_{BA}(t) &= \varphi_3(t)-\varphi_5(t)+D_6(t)
\end{aligned}$$

The input/output balance is defined by:

$$
\begin{aligned}
D_1(t) &= \frac{q_{in}(t)}{V}\left[X_{DCO,in}(t) - X_{DCO}(t)\right] \\
D_2(t) &= \frac{q_{in}(t)}{V}\left(-S_O(t)\right) + Kq_a(t)\left[S_{O,sat} - S_O(t)\right] \\
D_3(t) &= \frac{q_{in}(t)}{V}\left[S_{NH,in}(t) - S_{NH}(t)\right] \\
D_4(t) &= \frac{q_{in}(t)}{V}\left[-S_{NO}(t)\right] \\
D_5(t) &= \frac{q_{in}(t)}{V}\left[X_{BH,in}(t) - X_{BH}(t) + f_R\frac{1-f_W}{f_R+f_W}X_{BH}(t)\right] \\
D_6(t) &= \frac{q_{in}(t)}{V}\left[-X_{BA}(t) + f_R\frac{1-f_W}{f_R+f_W}X_{BA}(t)\right]
\end{aligned}
\tag{30}
$$

Let us first define the vector of measures, the control vector and the vector of unknown inputs in order to build a multiple model of the wastewater treatment process that will be used to apply the estimation methods proposed in this article.

In conformity with the hypothesis 1, the output vector is:

$$y(t) = \left[X_{DCO}(t), \quad S_O(t), \quad S_{NH}(t), \quad S_{NO}(t)\right]^T \tag{31}$$

the known input vector is:

$$u(t) = \left[X_{DCO,in}(t), \quad q_a(t)\right]^T \tag{32}$$

and the unknown input vector is:

$$d(t) = \left[S_{NH,in}(t), \quad X_{BH,in}(t)\right]^T \tag{33}$$

For numerical applications, the following heterotrophic growth and decay kinetic parameters are used [31]: $\mu_H = 3.733[1/24h]$, $\mu_A = 0.3[1/24h]$, $K_S = 20[g/m^3]$, $f_{SS} = 0.79$, $K_{OH} = 0.2[g/m^3]$, $K_{OA} = 0.4[g/m^3]$, $K_{NO} = 0.5[g/m^3]$, $K_{NH,A} = 1[g/m^3]$, $b_H = 0.3[1/24h]$, $b_A = 0.05[1/24h]$, $\eta_{NOg} = 0.8$.

In the following, variations of b_A and b_H around their nominal values will be considered as can be seen on figure 3. The stoichiometric parameters are $Y_H = 0.6$[g cell formed], $Y_A = 0.24$[g cell formed], $i_{XB} = 0.086$[g N in biomass], $i_{XP} = 0.06$[g N in endogenous mass], $f_P = 0.1$ and the oxygen saturation concentration is $S_{O,sat} = 10[g/m^3]$. The following numerical values are considered here for the fractions f_R and f_W: $f_R = 1.1$ and $f_W = 0.04$. The volume of the tank is $1333\,[m^3]$.

4.2. Multiple model design

A MM is built to design observers allowing state and unknown input estimation. Since the transformation of nonlinear system (29) into a MM form does not constitutes the main objective of the paper, and for lack of space, only the essential points (briefly described in section 2) are given in the following. For more details on this procedure the reader is referred to [23]. Considering the process equations (29), it is natural to define the following premise variables since they mainly contribute to the definitions of the nonlinearity of the system:

$$z_1(u(t)) = \frac{q_{in}(t)}{V} \tag{34a}$$

$$z_2(x(t), u(t)) = \frac{X_{DCO}(t)}{K_{DCO} + X_{DCO}(t)} \frac{S_O(t)}{K_{OH} + S_O(t)} \tag{34b}$$

$$z_3(x(t), u(t)) = \frac{X_{DCO}(t)}{K_{DCO} + X_{DCO}(t)} \frac{S_{NO}(t)}{K_{NO} + S_{NO}(t)} \frac{K_{OH}}{K_{OH} + S_O(t)} \tag{34c}$$

$$z_4(x(t), u(t)) = \frac{1}{K_{OA} + S_O(t)} \frac{S_{NH}(t)}{K_{NH,A} + S_{NH}(t)} X_{BA}(t) \tag{34d}$$

The system (29) can be written in a quasi-LPV form

$$\dot{x}(t) = A(x, u)x(t) + B(x, u)u(t) + E(x, u)d(t) \tag{35}$$

with matrices $A(x,u)$, $B(x,u)$ and $E(x,u)$ expressed by using the premise variables previously defined:

$$A(x,u) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & a_{15} & a_{16} \\ 0 & a_{22} & 0 & 0 & a_{25} & 0 \\ 0 & a_{32} & -z_1(u) & 0 & a_{35} & a_{36} \\ 0 & a_{42} & 0 & -z_1(u) & a_{45} & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & a_{62} & 0 & 0 & 0 & a_{66} \end{bmatrix} \tag{36}$$

$$B(u) = \begin{bmatrix} z_1(u) & 0 \\ 0 & K\,S_{O,sat} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad E(u) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ z_1(u) & 0 \\ 0 & 0 \\ 0 & z_1(u) \\ 0 & 0 \end{bmatrix} \tag{37}$$

where:

$$
\begin{aligned}
a_{11}(x,u) &= -z_1(x,u) \\
a_{15}(x,u) &= -\frac{\mu_H}{Y_H} z_2(x,u) + (1-f_P)\, b_H - \frac{\mu_H\, \eta_{NOg}}{Y_H} z_3(x,u) \\
a_{16}(x,u) &= (1-f_P)\, b_A \\
a_{22}(x,u) &= -z_1(x,u) - K\, q_a - \frac{4.57 - Y_A}{Y_A} \mu_A\, z_4(x,u) \\
a_{25}(x,u) &= \frac{(Y_H - 1)\mu_H}{Y_H} z_2(x,u) \\
a_{32}(x,u) &= -(i_{XB} + \frac{1}{Y_A})\mu_A\, z_4(x,u) \\
a_{35}(x,u) &= (i_{XB} - f_P\, i_{XP}) b_H - i_{XB}\, \mu_H\, z_2(x,u) - i_{XB}\, \mu_H\, \eta_{NOg}\, z_3(x,u) \\
a_{36}(x,u) &= (i_{XB} - f_P\, i_{XP})\, b_A \\
a_{42}(x,u) &= \frac{1}{Y_A} \mu_A z_4(x,u) \\
a_{45}(x,u) &= \frac{Y_H - 1}{2.86\, Y_H} \mu_H\, \eta_{NOg}\, z_3(x,u) \\
a_{55}(x,u) &= \mu_H\, z_2(x,u) - b_H + z_1(x,u) \left[\frac{f_W(1+f_R)}{f_R + f_W} - 1 \right] + \mu_H\, \eta_{NOg}\, z_3(x,u) \\
a_{62}(x,u) &= \mu_A\, z_4(x,u) \\
a_{66}(x,u) &= z_1(x,u) \left[\frac{f_W(1+f_R)}{f_R + f_W} - 1 \right] - b_A
\end{aligned} \qquad (38)
$$

The decomposition of the four premise variables (34) is realized by using the convex polytopic transformation (5). The scalars $z_{j,1}$ and $z_{j,2}$ are defined as in (6) and the functions $F_{j,1}(z_j(x,u))$ and $F_{j,2}(z_j(x,u))$ are given by (7) for $j = 1, ..., 4$. By multiplying the functions $F_{j,\sigma_i^j}(z_j(x,u))$, the $r = 16$ weighting functions $\mu_i(z(x,u))$ $(i = 1, \cdots, 16)$ are obtained:

$$
\mu_i(z(x,u)) = F_{1,\sigma_i^1}(z_1(u)) F_{2,\sigma_i^2}(z_2(x,u)) F_{3,\sigma_i^3}(z_3(x,u)) F_{4,\sigma_i^4}(z_4(x,u)) \qquad (39)
$$

The constant matrices A_i, B_i and E_i defining the 16 submodels, are determined by using the matrices A and B and the scalars z_{j,σ_i^j}:

$$
\begin{aligned}
A_i &= A(z_{1,\sigma_i^1}, z_{2,\sigma_i^2}, z_{3,\sigma_i^3}, z_{4,\sigma_i^4}) && \text{(40a)}\\
B_i &= B(z_{1,\sigma_i^1}) && \text{(40b)}\\
E_i &= E(z_{1,\sigma_i^1}) \qquad i = 1, ..., 16,\ j = 1, ..., 4 && \text{(40c)}
\end{aligned}
$$

Thus, the nonlinear model (29)-(31) is equivalently written under the multiple model form:

$$
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x, u)[A_i x(t) + B_i u(t) + E_i d(t)] && \text{(41a)}\\
y(t) &= Cx(t) && \text{(41b)}
\end{aligned}
$$

with

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{42}
$$

4.3. Uncertainties in the MM representation of the ASM1 model

The MM form used for the ASM1 model was previously proposed. Its structure is slightly modified in order to take into account parameter uncertainties of b_H and b_A. These parameters appear in the coefficients a_{15}, a_{16}, a_{35}, a_{36}, a_{55} and a_{66} in (38), that allow in (36) to separate the uncertain part $\Delta A(t)$ from the perfectly known part A. The variation of these parameters is 20% for b_H of its nominal value and for b_A 25% of its nominal value [32]. The uncertainties effect, taken into account in the matrices $A + \Delta A(t)$ can be written:

$$\Delta A(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.2\Delta b_H(t) & 0.25\Delta b_A(t) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2\Delta b_H(t) & 0.25\Delta b_A(t) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2\Delta b_H(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25\Delta b_A(t) \end{bmatrix} \quad (43)$$

Moreover the uncertain term is written under the form $\Delta A(t) = M^a F_a(t) N^a$ with the matrices:

$$M^a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; F_a(t) = \begin{bmatrix} 0.2\Delta b_H(t) & 0 \\ 0 & 0.25\Delta b_A(t) \end{bmatrix}, \; N^a = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $F_a(t)$ has the following property $F_a^T(t) F_a(t) \leq I$.

Thus, the equations (41) are modified as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x,u)[(A_i + \Delta A(t))\, x(t) + B_i\, u(t) + E_i\, d(t)] \quad (44a)$$

$$y(t) = C\, x(t) + G\, d(t) \quad (44b)$$

Let us see in the next section the state estimation results obtained by the proposed observer approach.

4.4. Proportional integral observer synthesis and simulation results

In the following, the observer approach described in section 3 is applied to the ASM1 model (29)-(31). The MM constructed in section 4.2 for the

ASM1 model is used to build the observer. A zero mean random signal $\delta(t)$ is added on the output to model noise measurements. The output is then given by:

$$y(t) = C\,x(t) + \delta(t) \tag{45}$$

The data used for simulation are generated with the complete ASM1 model ($n = 13$) [26, 33], in order to represent a realistic behavior of a WWTP. Even if the observer design is based on a MM form of the reduced ASM1 model ($n = 6$) it will be seen that the estimation results are satisfactory.

Let us consider the ASM1 model (29) under the equivalent MM form (44) with A_i, B_i, C, $\Delta A(t)$ defined by (40)-(42)-(43). Applying the Theorem 1, the observer (13) is designed by finding positive scalars ϵ_{1i}, ϵ_{2i} ($i = 1, ..., r$ with $r = 16$), positive definite matrices P_1 and P_2 and matrices $\bar{P}_j$ ($j = 1, \cdots, 16$) -that are not given here- such that the convergence conditions, given in Theorem 1 hold. The value of the attenuation rate from the known and unknown inputs $u(t)$ and $d(t)$ to the state and fault estimation error $e_a(t)$ is $\bar{\gamma} = 1.64$. The system inputs are represented in figure 2. The time varying parameters b_H and b_A are displayed in figure 3. A comparison between respectively the actual state variables, the unknown inputs and their respective estimates is depicted in the figures 4 and 5. The discrepancies near the time origin are only due to the arbitrarily choice of the initial conditions of the state observer. Despite of that choice, due to the convergence property of the observer, the reconstructed states fully represent the state of the process. Concerning the unknown inputs estimation (figure 5), the estimation errors in some time intervals are essentially due to the local fast variations of these variables, which is not in fully concordance with the hypothesis made for the

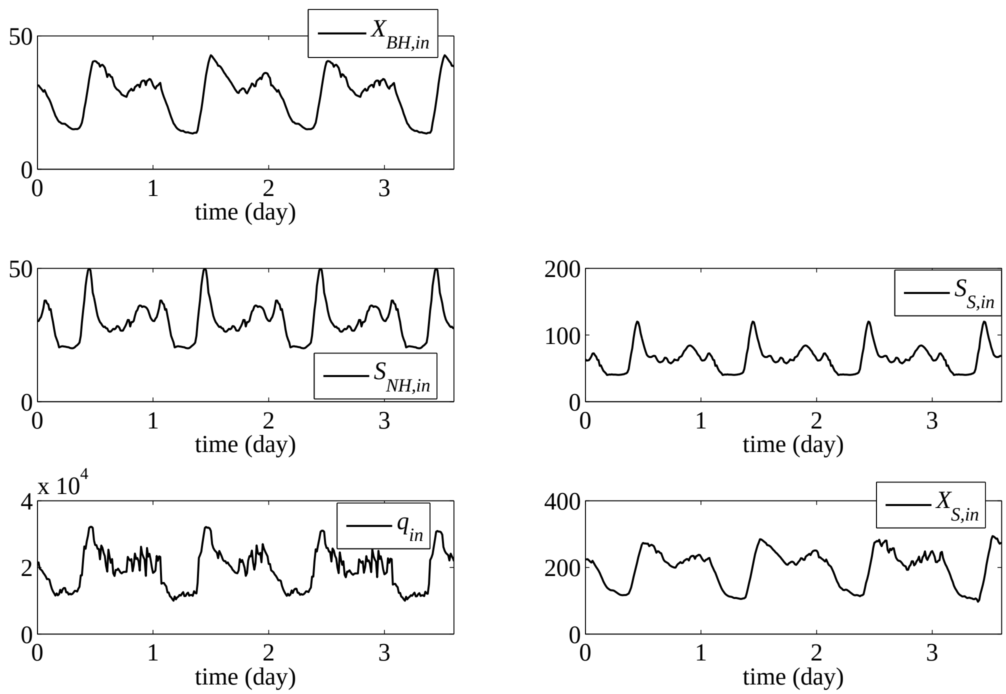

Figure 2: Inputs of wastewater treatment process

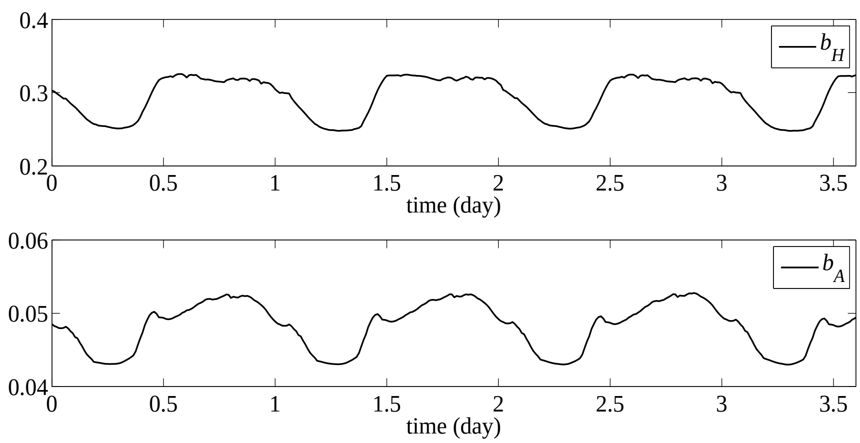

Figure 3: Varying parameters b_H and b_A

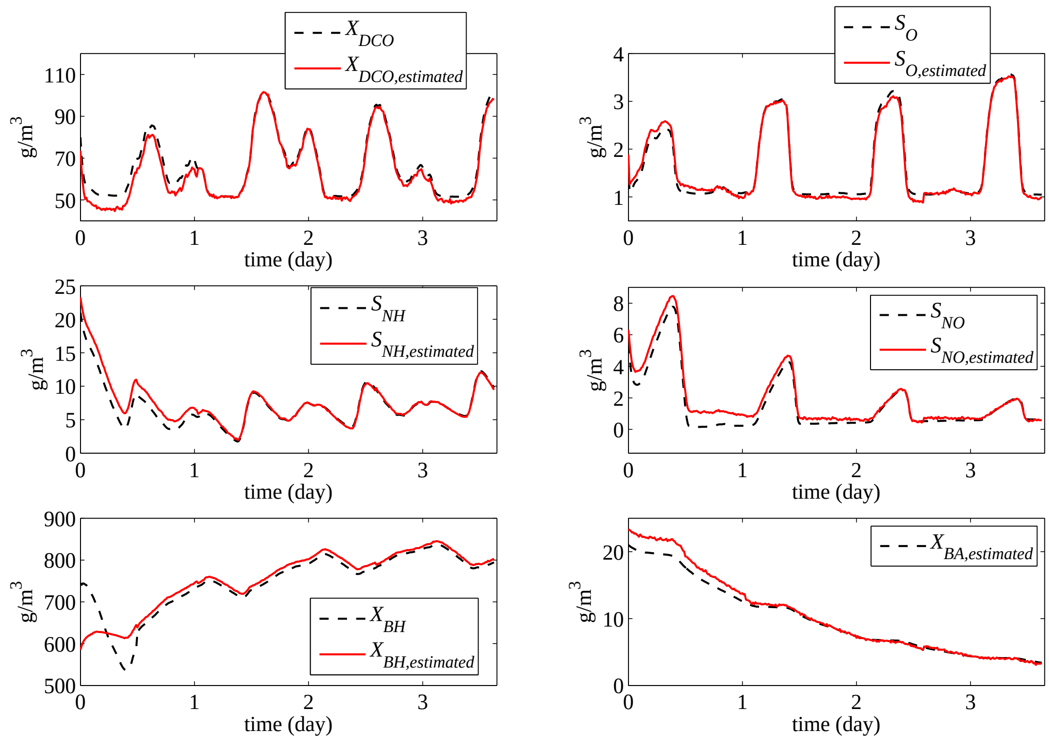

Figure 4: State estimation using the PIO with linear output

unknown inputs: $\dot{d} = 0$. However, the unknown input estimation remains globally correct. It should be highlighted that the input and output data are generated by the complete ASM1 model ($n = 13$) [26], whereas the model used for MM modeling and state estimation is the reduced one ($n = 6$).

5. Conclusion

The synthesis of a proportional integral observer adapted to uncertain nonlinear systems and affected by unknown inputs is proposed in this paper. The nonlinear system is equivalently represented by a multiple model with

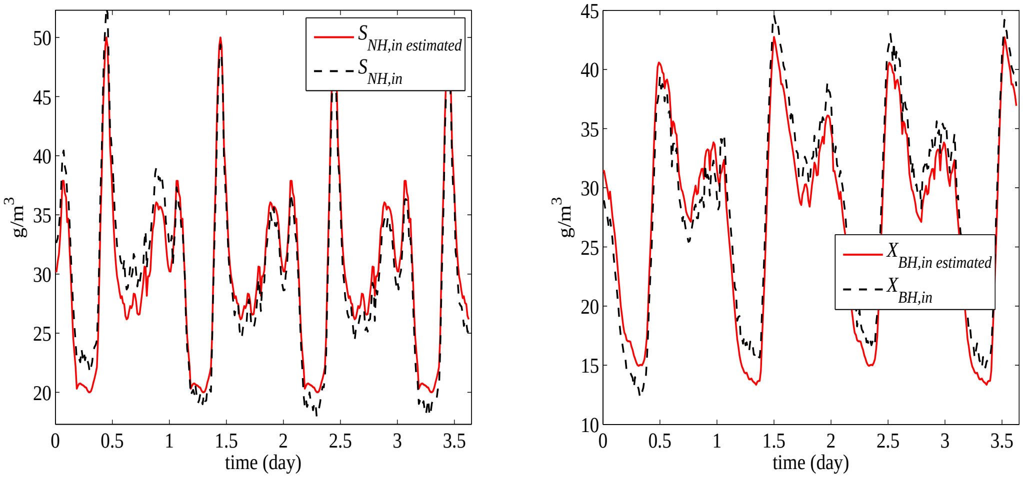

Figure 5: Unknown input estimation using the PIO

unmeasurable premise variables which is not intensively studied in literature since it could represent a difficult problem concerning the observer design and stability analysis studies. This theoretical points are then applied to a real process of a wastewater treatment plant that is characterized by parameter uncertainties and unknown inputs. The numerical simulation results for the proposed application show good state and unknown inputs estimation performances. As future prospects we can design dedicated observers by minimizing the $\mathcal{L}_2$ gain to the estimation error of a part of x and d (see a single element). An application to diagnosis based on the proposed observer design is in progress.

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Appendix A. Proof of Theorem 1

Proof. Let us define an augmented state and its estimate by $x_a(t) = \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}$ and $\hat{x}_a(t) = \begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix}$ respectively. The augmented state estimation error is defined by $e_a(t) = x_a(t) - \hat{x}_a(t)$. Using (10b) and (12), the system and observer equation can be written as

$$\dot{x}_a(t) = \sum_{i=1}^{r} \mu_i(x_a(t), u(t)) \left[(\overline{A}_i + \overline{\Delta A}_i(t)) x_a(t) + (\overline{B}_i + \overline{\Delta B}_i(t)) u(t) \right] \quad \text{(A.1a)}$$

$$y(t) = \overline{C} x_a(t) \quad \text{(A.1b)}$$

and

$$\dot{\hat{x}}_a(t) = \sum_{j=1}^{r} \mu_j(\hat{x}_a(t), u(t)) \left[\overline{A}_j \hat{x}_a(t) + \overline{B}_j u(t) + \overline{L}_j \left(y(t) - \hat{y}(t) \right) \right] \quad \text{(A.2a)}$$

$$\hat{y}(t) = \overline{C} \hat{x}_a(t) \quad \text{(A.2b)}$$

respectively, where

$$\overline{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \overline{\Delta A}_i(t) = \begin{bmatrix} \Delta A_i(t) & 0 \\ 0 & 0 \end{bmatrix} \quad \overline{\Delta B}_i(t) = \begin{bmatrix} \Delta B_i(t) \\ 0 \end{bmatrix}, \quad \overline{L}_i(t) = \begin{bmatrix} L_i^P \\ L_i^I \end{bmatrix}$$

One should note that in (A.1) the activating functions depend on $x_a(t)$, whereas they depend on $\hat{x}_a(t)$ in (A.2) and then the comparison of the state x_a (A.1a) and its reconstruction (A.2a) seem to be difficult. In order to cope with the difficulty of expressing the augmented state estimation error in a tractable way, (A.1a) is re-written, based on the property (3),

$$\dot{x}_a(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_a(t), u(t)) \mu_j(\hat{x}_a(t), u(t)) \left[(\overline{A}_j + \overline{A}_i - \overline{A}_j + \overline{\Delta A}_i(t)) x_a(t) \right.$$
$$\left. + (\overline{B}_j + \overline{B}_i - \overline{B}_j + \overline{\Delta B}_i(t)) u(t) \right] \quad \text{(A.3)}$$

Consequently, the augmented state estimation error obeys to the following nonlinear system

$$\begin{bmatrix} \dot{e}_a(t) \\ \dot{x}(t) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(x_a(t), u(t))\mu_j(\hat{x}_a(t), u(t)) \left\{ \begin{bmatrix} \overline{A}_j - \overline{L}_j\overline{C} & \tilde{A}_i - \tilde{A}_j + \widetilde{\Delta A}_i(t) \\ 0 & A_i + \Delta A_i(t) \end{bmatrix} \right.$$

$$\left. \cdot \begin{bmatrix} e_a(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} \overline{\Delta B}_i(t) & \tilde{E}_i - \tilde{E}_j \\ B_i + \Delta B_i(t) & E_i \end{bmatrix} \begin{bmatrix} u(t) \\ d(t) \end{bmatrix} \right\} \quad \text{(A.4a)}$$

$$e_a(t) = \begin{bmatrix} I_{n+n_d} & 0 \end{bmatrix} \begin{bmatrix} e_a(t) \\ x(t) \end{bmatrix} \quad \text{(A.4b)}$$

where $\widetilde{\Delta A}(t)_i = \begin{bmatrix} \Delta A_i(t) \\ 0 \end{bmatrix}$. Let $V(e_a(t), x(t))$ denote the following candidate Lyapunov function

$$V(x_a(t), x(t)) = \begin{bmatrix} e_a(t) \\ x(t) \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} e_a(t) \\ x(t) \end{bmatrix} \quad \text{(A.5)}$$

where P_1 and P_2 are symmetric positive definite matrices. The objective is to find the gains of the observer that minimize the $\mathcal{L}_2$-gain from the known and unknown inputs $u(t)$ and $d(t)$ to the state and fault estimation error $e_a(t)$. It is well known [38] that the $\mathcal{L}_2$-gain from $\begin{bmatrix} u(t) \\ d(t) \end{bmatrix}$ to $e_a(t)$ is bounded by γ if

$$\dot{V}(e_a(t), x(t)) + e_a^T(t)e_a(t) - \gamma^2(u^T(t)u(t) + d^T(t)d(t)) < 0 \quad \text{(A.6)}$$

Considering the Lyapunov function (A.5) and the trajectory of $e_a(t)$ defined by (A.4), the inequality (A.6) can be written as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(x_a(t))\mu_j(\hat{x}_a(t))\begin{bmatrix} e_a(t)\\ x(t)\\ u(t)\\ d(t)\end{bmatrix}^T\begin{bmatrix}\Psi_{ij}^{11} & \Phi_{ij}^{12} & \Phi_{ij}^{13} & \Psi_{ij}^{14}\\ * & \Phi_{ij}^{22} & \Phi_{ij}^{23} & P_2E_i\\ * & * & -\gamma^2 I_{n_u} & 0\\ * & * & * & -\gamma^2 I_{n_d}\end{bmatrix}\begin{bmatrix} e_a(t)\\ x(t)\\ u(t)\\ d(t)\end{bmatrix}<0 \tag{A.7}$$

where

$$\begin{aligned}
\Psi_{ij}^{11} &= \mathbb{S}(P_1\overline{A}_j - P_1\overline{L}_j\overline{C}) + I, & \Phi_{ij}^{12} &= P_1(\tilde{A}_i - \tilde{A}_j + \tilde{\Delta A}_i(t)),\\
\Phi_{ij}^{13} &= P_1\overline{\Delta B}_i(t), & \Psi_{ij}^{14} &= P_1(\tilde{E}_i - \tilde{E}_j),\\
\Phi_{ij}^{22} &= \mathbb{S}(P_2A_i + P_2\Delta A_i(t)), & \Phi_{ij}^{23} &= P_2(B_i + \Delta B_i(t))
\end{aligned}$$

Isolating the time varying entries $\Delta A_i(t)$, $\Delta B_i(t)$, $\widetilde{\Delta A}_i(t)$ and $\widetilde{\Delta B}_i(t)$, (A.7) becomes

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(x_a(t))\mu_j(\hat{x}_a(t))\begin{bmatrix} e_a(t)\\ x(t)\\ u(t)\\ d(t)\end{bmatrix}^T\left\{\begin{bmatrix}\Psi_{ij}^{11} & \Psi_{ij}^{12} & 0 & \Psi_{ij}^{14}\\ * & \Psi_{ij}^{22} & \Psi_{ij}^{23} & P_2E_i\\ * & * & -\gamma^2 I_{n_u} & 0\\ * & * & * & -\gamma^2 I_{n_d}\end{bmatrix}\right.$$

$$\left.+\mathbb{S}\left(\begin{bmatrix} P_1\overline{M}_i^a\\ P_2M_i^a\\ 0\\ 0\end{bmatrix}F_a(t)\begin{bmatrix} 0\\ N_i^{aT}\\ 0\\ 0\end{bmatrix}^T+\begin{bmatrix} P_1\overline{M}_i^b\\ P_2M_i^b\\ 0\\ 0\end{bmatrix}F_b(t)\begin{bmatrix} 0\\ 0\\ N_i^{bT}\\ 0\end{bmatrix}^T\right)\right\}\begin{bmatrix} e_a(t)\\ x(t)\\ u(t)\\ d(t)\end{bmatrix}<0 \tag{A.8}$$

where

$$\Psi_{ij}^{12} = P_1(\tilde{A}_i - \tilde{A}_j), \qquad \Psi_{ij}^{22} = \mathbb{S}(P_2A_i), \qquad \Psi_{ij}^{23} = P_2B_i$$

It is known that for any matrices X, Y and $F(t)$ of appropriate dimensions satisfying $F(t)^T F(t) \leq I$ and any positive scalar ε the following holds

$$XF(t)Y^T + YF(t)^T X^T \leq \varepsilon XX^T + \epsilon^{-1} YY^T \tag{A.9}$$

With the previous inequality (A.9) and the property of $F_a(t)$ and $F_b(t)$ given by (11), it can be stated that the LMI (A.8) is satisfied if the following holds

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(x_a(t))\mu_j(\hat{x}_a(t))\left\{\begin{bmatrix}\Psi_{ij}^{11} & \Psi_{ij}^{12} & 0 & \Psi_{ij}^{14}\\ * & \Psi_{ij}^{22} & \Psi_{ij}^{23} & P_2E_i\\ * & * & -\gamma^2 I_{n_u} & 0\\ * & * & 0 & -\gamma^2 I_{n_d}\end{bmatrix}\right.$$
$$+\mathbb{S}\left(\varepsilon_{i1}^{-1}\begin{bmatrix}P_1\overline{M}_i^a\\ P_2M_i^a\\ 0\\ 0\end{bmatrix}\begin{bmatrix}P_1\overline{M}_i^a\\ P_2M_i^a\\ 0\\ 0\end{bmatrix}^T + \varepsilon_{2i}^{-1}\begin{bmatrix}P_1\overline{M}_i^b\\ P_2M_i^b\\ 0\\ 0\end{bmatrix}\begin{bmatrix}P_1\overline{M}_i^b\\ P_2M_i^b\\ 0\\ 0\end{bmatrix}^T\right.$$
$$\left.\left.+\,\varepsilon_{1i}\begin{bmatrix}0\\ N_i^{aT}\\ 0\\ 0\end{bmatrix}\begin{bmatrix}0\\ N_i^{aT}\\ 0\\ 0\end{bmatrix}^T + \varepsilon_{2i}\begin{bmatrix}0\\ 0\\ N_i^{bT}\\ 0\end{bmatrix}\begin{bmatrix}0\\ 0\\ N_i^{bT}\\ 0\end{bmatrix}^T\right)\right\} < 0 \tag{A.10}$$

With some Schur complements and defining $\overline{P}_j = P_1\overline{L}_j$ and $\overline{\gamma} = \gamma^2$, the previous inequality becomes

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(x_a(t))\mu_j(\hat{x}_a(t))\mathcal{M}_{ij} < 0 \tag{A.11}$$

It follows that (A.6) is satisfied if the LMI (14) holds, which achieves the proof. □

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