New fault tolerant control strategy for nonlinear systems with multiple model approach

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***Abstract*— This paper addresses a new methodology to construct a fault tolerant control (FTC) in order to compensate actuator faults in nonlinear systems. This approach is based on the representation of the nonlinear model with a multiple model under Takagi-Sugeno's form. The proposed control requires a simultaneous estimation of the system states and of the occurring actuator faults. The performance of the control depends on the quality of the estimations, indeed, it is important to estimate accurately and rapidly the states and the faults. This task is then performed with an Adaptive Fast State and Fault Observer (AFSFO). The stability conditions are established with Lyapunov theory and expressed in linear matrix inequality formulation to ease the design of the FTC. Furthermore, relaxed stability conditions are given with the use of the Polya's theorem.**

***Index Terms*— Nonlinear systems, Takagi-Sugeno model (T-S), linear matrix inequality (LMI), Polya's theorem, Lyapunov theory, input-to-state stability (ISS), fault tolerant control (FTC)**

I. INTRODUCTION

Since several years, the problem of fault tolerance has been treated from many points of view. Two classes can then be considered: passive control and active control. The first class may be viewed as a robust control. It requires the a priori knowledge of the possible faults which may affect the system. The control is then designed in order to compensate them. The interest of this approach is the fact that no on line information is needed and the structure of the control law remains unchanged. The principal idea of this kind of control is based on the consideration of all possible faults as uncertainties which are taken into account for the design of the tolerant control by using different techniques like $\mathcal{H}_\infty$ [15], [13]. Generally, the structure of the uncertainties (faults) are not taken into account in order to lead to a convex optimization problem. Furthermore, the class of considered faults is limited, it becomes then risky to use only the passive fault tolerant control (see [11] for more details).

The second class concerns active fault tolerant control which is more interesting due to its possibility to take into account a large class of faults, because of its variable structure which may change in the presence of faults. The knowledge of some informations about these last are required and are obtained from a Fault Detection and Diagnosis (FDD) block. Different ideas are developed in the literature, for example, a Control Law Re-scheduling [9], [7], [17]. This approach requires a very robust Fault Detection and Isolation (FDI) block which constitutes its major disadvantage. Indeed, a false alarm or a non detected fault can lead to degraded performance or even to instability. Other smooth fault tolerant control laws are proposed in [5] for Takagi-Sugeno systems and in [14] for LPV systems.

Many efforts are dedicated to the problem of designing an active fault tolerant control of nonlinear systems, among them, the use of Takagi-Sugeno representation that combines simplicity and accuracy of nonlinear behaviors, it is introduced initially in [18]. The idea is to consider a set of linear sub-systems. An interpolation of all these sub-models with nonlinear functions satisfying the convex sum property allows to obtain the global behavior of the system described in a large operating range. Some works can be mentioned in the FTC field for nonlinear systems. For example, in [4], the authors took into account actuator faults for nonlinear descriptor systems with Lipschitz nonlinearities. In [9], a method which requires only the fault isolation was proposed for T-S systems. It was based on a bank of observer based controllers. A switching mechanism is then designed depending on the obtained residuals. More recently, in [5], the FTC strategy with trajectory tracking and proportional-integral observer (PIO), is developed for the T-S systems with weighting functions depending on the state of the system which is not accessible for measure.

II. TAKAGI-SUGENO STRUCTURE FOR MODELING

The T-S modeling allows to represent the behavior of nonlinear systems by the interpolation of a set of linear sub-models. Each sub-model contributes to the global behavior of the nonlinear system through a weighting function $\mu_i(\xi(t))$. The T-S structure is given by

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the input vector, $y(t) \in \mathbb{R}^{n_y}$ represents the output vector. $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times n_u}$ and $C \in \mathbb{R}^{n_y \times n}$ are known matrices. The functions $\mu_i(\xi(t))$ are the weighting functions depending on the variable $\xi(t)$ which is accessible for measure (as the input or the output of the system). These functions verify the following properties

$$\begin{cases} \sum_{i=1}^{r} \mu_i(\xi(t)) = 1 \\ 0 \leq \mu_i(\xi(t)) \leq 1 \quad \forall i \in \{1, 2, ..., r\} \end{cases} \tag{2}$$

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Obtaining a T-S model (1) can be performed from different methods such as linearization around some operating points and using adequate weighting functions. It can also be obtained by black-box approaches which allow to identify the parameters of the model from input-output data. Finally, a T-S model can be obtained from the well-known nonlinear sector transformations [19], [12]. This transformation allows to obtain an exact T-S representation of nonlinear model with no information loss on a compact set of the state space.

Thanks to the convex sum property of the weighing functions (2), it is possible to generalize some tools developed in the linear domain to the nonlinear systems. This representation (1) is very interesting in the sense that it simplifies the stability studies of nonlinear systems and the design of control laws and observers. In [19], [6], the stability and stabilization tools are inspired from the study of linear systems. In [1], [10], the authors worked on the problem of state estimation and diagnosis of T-S fuzzy systems. The proposed approaches in these last papers rely on the generalization of the classical observers (Luenberger Observer [8] and Unknown Input Observer (UIO) [3]) to the nonlinear systems. Recently in [16], a new approach, derived from the Polya's theorem, leads to asymptotic necessary and sufficient stability conditions.

In the remaining of the paper, the two following lemmas are used.

Lemma 1: Consider two matrices X and Y with appropriate dimensions and G a positive definite matrix. The following property is verified

$$X^TY + Y^TX \leq X^TGX + Y^TG^{-1}Y \quad G > 0 \tag{3}$$

Lemma 2: (Congruence) Consider two matrices P and Q, if P is positive definite and if Q is a full column rank matrix, then the matrix QPQ^T is positive definite.

$\lambda_{max}(M)$ represents the maximum singular value of the matrix M.

III. Problem Statement

Under actuator faults, the system (1) can be re-written in the following form

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{r} \mu_i(\xi(t)) \left(A_i x(t) + B_i \left(u(t) + f(t)\right)\right) \\ y(t) = Cx(t) \end{cases} \tag{4}$$

where $f(t)$ is an actuator fault. Faults can affect a system in several different ways. They can be represented by an additive or a multiplicative external signal. In this case, they affect the performances of the system but its stability is not affected. It can be pointed out that if the fault depends on the system state, it can change the structure of the model and cause its unstability. For instance, malfunctions of the actuator can be represented by a faulty control input defined by $u_f(t) = (I_{n_u} - \gamma)u(t)$ which can be easily re-written in the form of an external additive signal: $(u(t) + f(t))$ where $f(t) = -\gamma u(t)$ and $\gamma = diag\left(\gamma_1, \gamma_2, \quad \cdots \gamma_{n_u}\right), 0 \leq \gamma_i \leq 1$ $(i = 1, ..., n_u)$ where

$$\begin{cases} \gamma_i = 1 \Rightarrow \text{total failure of the } i^{th} \text{ actuator} \\ \gamma_i = 0 \Rightarrow \text{the } i^{th} \text{ actuator is healthy} \\ \gamma_i \in \left]\ 0 \quad 1\ \right[\Rightarrow \text{loss of effectiveness of the } i^{th} \text{ actuator} \end{cases}$$

For example if $\gamma_2 = 0.4$, there is a 40% loss of effectiveness of the second actuator. Note that such multiplicative faults can cause the system instability.

Assumption 1: In this paper, it is assumed that

- **A1.** the faults are assumed to have norm bounded first time derivative
$$\left\| \dot{f}(t) \right\| \leq f_{1max}, \quad 0 \leq f_{1max} < \infty \tag{5}$$
- **A2.** $rank(CB_i) = n_u$
- **A3.** Total actuator failures are not considered, i.e. $\gamma_i \in [0 \quad 1[$

In this paper, a new actuator fault tolerant control is proposed. Using a fast adaptive observer proposed in [20] and extended here to nonlinear T-S systems, the state and the fault affecting the system are estimated rapidly. The use of such an observer is motivated by the fact that if a fault occurs, it is important to detect it quickly and with a good accuracy in order to take it into account and preserve the system performances. With the use of Lyapunov theory, sufficient conditions are obtained for asymptotic stability in the constant fault case and for input-to-state stability (ISS) in the case of time varying faults. The LMI formulation is used for representing the obtained stability conditions in an adequate form for existing LMI solvers. Finally, relaxed stability conditions are obtained with the use of Polya's theorem [16].

IV. Fault tolerant control for nonlinear systems

An adaptive observer estimating the state and the faults of the system (4) is given by

$$\begin{cases} \dot{\hat{x}}(t) = \sum\limits_{i=1}^{r} \mu_i(\xi(t))(A_i\hat{x}(t) + B_i(u(t) + \hat{f}(t)) + L_i e_y(t)) \\ \hat{y}(t) = C\hat{x}(t) \\ \dot{\hat{f}}(t) = \Gamma \sum\limits_{i=1}^{r} \mu_i(\xi(t)) F_i(\dot{e}_y(t) + \sigma e_y(t)) \\ e_y(t) = y(t) - \hat{y}(t) \end{cases} \tag{6}$$

and the proposed active fault tolerant control takes the form

$$u(t) = -\sum_{i=1}^{r} \mu_i(\xi(t)) K_i \hat{x}(t) - \hat{f}(t) \tag{7}$$

The objective is to determine the parameters L_i, Γ, F_i, K_i and σ such that the state of the system converges asymptotically to zero if the fault $f(t)$ is constant or to a small ball around the origin in the case where $f(t)$ is time varying with norm bounded first time derivative. The expression describing the dynamic of the fault $f(t)$ given in (6) depends on both the output error and the derivative of the output error.

Let us consider the fault and state estimation errors $e_x(t) = x(t) - \hat{x}(t)$ and $e_f(t) = f(t) - \hat{f}(t)$. The dynamics of the state estimation error and the closed-loop system with the control (7) obey to the differential equations

$$\dot{e}_x(t) = \sum_{i=1}^{r} \mu_i(\xi(t)) \left(\Phi_i e_x(t) + B_i e_f(t)\right) \tag{8}$$

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t)) \left(\Xi_{ij} x(t) + B_i e_f + B_i K_j e_x\right) \tag{9}$$

where $\Phi_i = A_i - L_i C$ and $\Xi_{ij} = A_i - B_i K_j$.

Theorem 1: Under the assumptions 1, given positive scalars σ and β, if there exists symmetric and positive definite matrices $\mathcal{X} \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n_f\times n_f}$ (with $n_f = n_u$) and matrices $M_i \in \mathbb{R}^{n_u\times n}$ and $N_i \in \mathbb{R}^{n\times n_y}$ and a positive scalar η solution to the optimization problem

$$\min \quad \eta \quad s.t. \tag{10}$$

$$\begin{pmatrix} \eta I & B_i^T P_2 - F_i C \\ \left(B_i^T P_2 - F_i C\right)^T & \eta I \end{pmatrix} > 0 \tag{11}$$

$$\mathcal{Q}_{ij} = \begin{pmatrix} S_{ij} & B_i M_j & B_i & 0 & 0 \\ * & -2\beta\mathcal{X} & 0 & \beta I & 0 \\ * & * & -2\beta I & 0 & \beta I \\ * & * & * & \Omega_j & \mathcal{R}_{ij} \\ * & * & * & * & \Psi_{ij} \end{pmatrix} < 0 \tag{12}$$

$$S_{ij} = XA_i^T + XA_i - B_i M_i - M_i^T B_i^T \tag{13}$$

$$\Omega_j = A_i^T P_2 + P_2 A_i - N_i C - C^T N_i^T \tag{14}$$

$$\mathcal{R}_{ij} = -\frac{1}{\sigma}(A_j^T P_2 - C^T N_j^T) B_i \tag{15}$$

$$\Psi_{ij} = -\frac{1}{\sigma}\left(B_i^T P_2 B_j + B_j^T P_2 B_i\right) + \frac{1}{\sigma} G \tag{16}$$

then the state of the system $x(t)$, the state estimation error and the fault estimation error $e_f(t)$ are bounded. Furthermore, if $f_{1max} = 0$, these variables converge asymptotically to zero. The gains of the observer and the fault tolerant control are given by F_i, $L_i = P_2^{-1} N_i$ and $K_i = M_i \mathcal{X}^{-1}$.

Proof: In order to prove both the stability of the closed-loop system and the convergence of the state and fault estimation errors, the proof is based on a Lyapunov function depending on $x(t)$, $e_x(t)$ and $e_f(t)$ defined by

$$V(t) = x^T(t) P_1 x(t) + e_x^T(t) P_2 e_x(t) + \frac{1}{\sigma} e_f(t) \Gamma^{-1} e_f(t) \tag{17}$$

According to the equations (8) and (9), the time derivative of $V(t)$ is given by

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))(x^T(t)\Pi_{ij} x(t) \\ &+ e_x^T(t)\Omega_i e_x(t) + 2x^T(t) P_1 B_i K_j e_x(t) \\ &+ 2x^T(t) P_1 B_i e_f(t) + 2e_x^T(t) P_2 B_i e_f(t) \\ &+ \frac{2}{\sigma} e_f^T(t) \Gamma^{-1} \dot{e}_f(t)) \end{aligned} \tag{18}$$

where $\Pi_{ij} = \Xi_{ij}^T P_1 + P_1 \Xi_{ij}$ and $\Omega_i = \Phi_i^T P_2 + P_2 \Phi_i$. Knowing that $\dot{e}_f(t) = \dot{f}(t) - \dot{\hat{f}}(t)$ and given the expression of $\hat{f}(t)$ in (6), one obtains

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))(x^T(t)\Pi_{ij} x(t) \\ &+ e_x^T(t)\Omega_i e_x(t) + 2x^T(t) P_1 B_i K_j e_x(t) \\ &+ 2x^T(t) P_1 B_i e_f(t) + 2e_x^T(t) P_2 B_i e_f(t) \\ &- \frac{2}{\sigma} e_f^T(t) F_i(\dot{e}_y(t) + \sigma e_y(t)) + \frac{2}{\sigma} e_f^T(t) \Gamma^{-1} \dot{f}(t)) \end{aligned} \tag{19}$$

Using the differential equation (8) generating $e_x(t)$, the following is obtained

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))(x^T(t)\Pi_{ij} x(t) \\ &+ e_x^T(t)\Omega_i e_x(t) + 2x^T(t) P_1 B_i K_j e_x(t) \\ &+ 2x^T(t) P_1 B_i e_f(t) + 2e_x^T(t) P_2 B_i e_f(t) \\ &- \frac{2}{\sigma} e_f^T(t) F_i C \Phi_j e_x(t) - \frac{2}{\sigma} e_f^T(t) F_i C B_j e_f(t) \\ &- 2e_f^T(t) F_i C e_x(t) + \frac{2}{\sigma} e_f^T(t) \Gamma^{-1} \dot{f}(t)) \end{aligned} \tag{20}$$

Using Lemma 1 and assumption **A1**, we deduce that

$$\begin{aligned} & 2\frac{1}{\sigma} e_f^T(t) \Gamma^{-1} \dot{f}(t) \\ \leq\ & \frac{1}{\sigma} e_f^T(t) G e_f + \frac{1}{\sigma} \dot{f}^T(t) \Gamma^{-1} G^{-1} \Gamma^{-1} \dot{f}(t) \\ \leq\ & \frac{1}{\sigma} e_f^T(t) G e_f + \frac{1}{\sigma} f_{1\max}^2 \lambda_{max}\left(\Gamma^{-1} G^{-1} \Gamma^{-1}\right) \end{aligned} \tag{21}$$

and using assumption **A2**, it is possible to obtain F_i and P_2 such that $B_i^T P_2 = F_i C$ holds. The time derivative of the Lyapunov function (20) is bounded as follows

$$\dot{V}(t) \leq \tilde{x}^T(t) \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t)) \Delta_{ij} \tilde{x}(t) + \delta \tag{22}$$

where $\tilde{x}^T(t) = (x^T(t) \quad e_x^T(t) \quad e_f^T(t))^T$, $\delta = \frac{1}{\sigma} f_{1\max}^2 \lambda_{\max}\left(\Gamma^{-1} G^{-1} \Gamma^{-1}\right)$ and

$$\Delta_{ij} = \begin{pmatrix} \Pi_{ij} & P_1 B_i K_j & P_1 B_i \\ * & \Omega_i & -\frac{1}{\sigma}\Phi_j^T P_2 B_i \\ * & * & \Psi_{ij} \end{pmatrix} \tag{23}$$

If the following equation holds

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t)) \Delta_{ij} < 0 \tag{24}$$

it is established that

$$\dot{V}(t) < -\varepsilon \left\|\tilde{x}(t)\right\|^2 + \delta \tag{25}$$

where ε is defined by

$$\varepsilon = \lambda_{min}\left(-\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t)) \Delta_{ij}\right) \tag{26}$$

It follows that $\dot{V}(t) < 0$ if $\varepsilon \|\tilde{x}(t)\|^2 > \delta$, and according to Lyapunov stability theory the state $x(t)$, the state estimation error $e_x(t)$ and the fault estimation error $e_f(t)$ converge to a small ball of convergence around the origin. This ball is smaller as the constant δ converges to zero.

In order to achieve the proof, it remains to establish some LMI conditions to ensure that (24) and $B_i^T P_2 = F_i C$ holds. The latter is first considered.

As pointed out in [20], it is difficult to solve simultaneously, the inequality $\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\Delta_{ij} < 0$ with the equality constraint $B_i^T P_2 = F_i C$. A technique for reducing this difficulty is to formulate the equality constraint as an optimization problem [2]

$$\min \;\eta \;\; s.t. \;\; \begin{pmatrix} \eta I & B_i^T P_2 - F_i C \\ \left(B_i^T P_2 - F_i C\right)^T & \eta I \end{pmatrix} > 0 \tag{27}$$

For the sake of simplicity, the following notations will be used

$$Y_\xi = \sum_{i=1}^{r} \mu_i(\xi(t)) Y_i, \;\; Y_{\xi\xi} = \sum_{i=1}^{r}\sum_{i=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t)) Y_{ij} \tag{28}$$

where Y_i and Y_{ij} are given matrices. Using this representation, the inequality (24) becomes

$$\Delta_{\xi\xi} = \begin{pmatrix} \Pi_{\xi\xi} & \Theta_{\xi\xi} \\ \Theta_{\xi\xi}^T & \Lambda_{\xi\xi} \end{pmatrix} < 0 \tag{29}$$

where

$$\Theta_{ij} = \begin{pmatrix} P_1 B_i K_j & P_1 B_i \end{pmatrix} \tag{30}$$

$$\Lambda_{ij} = \begin{pmatrix} \Omega_i & -\frac{1}{\sigma}\Phi_j^T P_2 B_i \\ -\frac{1}{\sigma}\left(\Phi_j^T P_2 B_i\right)^T & \Psi_{ij} \end{pmatrix} \tag{31}$$

Consider a matrix X defined as follows

$$X = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & X_1 \end{pmatrix}, \;\; X_1 = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & I \end{pmatrix} \tag{32}$$

Using Lemma 2, post and pre-multiplying the inequality (29) by X, it follows that (29) is equivalent to the following inequality

$$\begin{pmatrix} P_1^{-1}\Pi_{\xi\xi}P_1^{-1} & P_1^{-1}\Theta_{\xi\xi}X_1 \\ X_1\Theta_{\xi\xi}^T P_1^{-1} & X_1\Lambda_{\xi\xi}X_1 \end{pmatrix} < 0 \tag{33}$$

Since the following inequality holds

$$\begin{aligned} &\left(X_1 + \beta\Lambda_{\xi\xi}^{-1}\right)^T \Lambda_{\xi\xi}\left(X_1 + \beta\Lambda_{\xi\xi}^{-1}\right) \leq 0 \\ \Leftrightarrow \quad & X_1\Lambda_{\xi\xi}X_1 \leq -\beta\left(X_1 + X_1^T\right) - \beta^2\Lambda_{\xi\xi}^{-1} \end{aligned} \tag{34}$$

and with a Schur complement, it follows that the inequality (33) holds if (35), displayed below, is satisfied

$$\begin{pmatrix} P_1^{-1}\Pi_{\xi\xi}P_1^{-1} & P_1^{-1}\Theta_{\xi\xi}X_1 & 0 \\ \Theta_{\xi\xi}^T P_1^{-1}X_1 & -2\beta X_1 & \beta I \\ 0 & \beta I & \Lambda_{\xi\xi} \end{pmatrix} < 0 \tag{35}$$

Using the notations (28) and the definitions of the matrices $\Pi_{\xi\xi}$, $\Theta_{\xi\xi}$ and $\Omega_{\xi\xi}$ given in the equalities (30) and (31), and with the changes of variables $\mathcal{X} = P_1^{-1}$, $M_i = K_i\mathcal{X}$, $N_i = P_2 L_i$ it is easy to obtain the inequalities given in the theorem 1. Finally, the inequality (25) is satisfied, if the optimization problem given by (10) under LMI constraints (12) has a solution, which ends the proof. ■

Remark 1: Note that if the fault $f(t)$ is constant, then $f_{1max} = 0$ and $\delta = 0$, consequently the asymptotic stability is achieved, since $\dot{V}(t) < 0$ for every $\tilde{x}(t)$.

Remark 2: After solving the optimization problem given in the theorem 1, the input-to-state stability condition given in (25) is satisfied. Thus, in the case of time varying faults with bounded first time derivative, the state $x(t)$, the state estimation error $e_x(t)$ and the fault estimation error $e_f(t)$ converge to an origin centered ball defined by the terms δ and ε. Choosing a high value for the parameter Γ will minimize δ without changing ε (that does not depend on Γ) and consequently will minimize the radius of the ball in which $\tilde{x}$ converges. It thus improves the accuracy of the estimation.

Remark 3: The objective of fault tolerant control is to compensate the faults, so it is important to estimate them as soon as possible with a good accuracy. The adaptive observer studied in this paper can be considered as an improvement of the classical PI observer, in the sense that convergence of the state and fault estimations is proved (in an origine centered ball) even in non constant fault case, whereas the assumption of constant fault is needed to prove the estimation error convergence when using a PI observer.

V. Conservatism reduction with Polya's theorem

In the previous section, the proposed result may be conservative in the sense that common Lyapunov matrices were sought to satisfy r^2 LMIs. Recently, a new interesting method to reduce the conservativeness of the matrix summations inequality has been proposed with the use of Polya's theorem [16].

Let us consider the inequality (36)

$$\Delta_{\xi\xi} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\Delta_{ij} < 0 \tag{36}$$

where Δ_{ij} is defined in equation (23). Knowing that $\left(\sum_{i=1}^{r}\mu_i(\xi(t))\right)^p = \sum_{i=1}^{r}\mu_i(\xi(t)) = 1$ where p is a positive integer, the inequality (36) is equivalent to

$$\left(\sum_{i=1}^{r}\mu_i(\xi(t))\right)^p \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(\xi(t))\mu_j(\xi(t))\Delta_{ij} < 0 \tag{37}$$

1) example: In order to better assimilate this approach, let us consider this example where $r = 2$ (two sub-models), then the stability is ensured if inequality (36) holds. Classically, the negativity of (36) is ensured if all the terms Δ_{ij} are negative for $i, j = 1, 2$. However, using Polya's theorem, the negativity of the inequality (36) is equivalent to the negativity of (37). Choosing $p = 1$ we obtain three summations, and the inequality (37) is equivalent to

$$\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2} \mu_{i_1}\mu_{i_2}\mu_{i_3}\Delta_{i_1 i_2} < 0 \tag{38}$$

Consequently, the negativity of (36) is ensured if

$$\Delta_{11} < 0, \quad \Delta_{22} < 0 \tag{39}$$
$$\Delta_{11} + \Delta_{12} + \Delta_{21} < 0 \tag{40}$$
$$\Delta_{22} + \Delta_{21} + \Delta_{12} < 0 \tag{41}$$

By comparison to the classical result where all inequalities $\Delta_{ij} < 0$ for all $i, j = 1, 2$ are needed, this new approach only requires the negativity of the terms Δ_{ij} and the negativity of the terms Δ_{ij} for $i \neq j$ is no longer needed.

As explained in [16], the negativity of (36) is guaranteed if inequality (37) is verified with a given parameter p. Increasing p provides less conservative stability conditions and if $p \to +\infty$ asymptotic necessary and sufficient conditions for the negativity of (36) are obtained. The authors proposed also an algorithm to compute finite values of p which gives necessary and sufficient conditions with a given accuracy. The reader can refer to the paper [16] for more details on Polya's theorem based relaxation approach.

In order to reduce the conservatism introduced to ensure (1), the Polya's theorem is applied directly on the inequality (35), with the changes of variables $\mathcal{X} = P_1^{-1}$, $M_i = K_i\mathcal{X}$, $N_i = P_2 L_i$, for a suitable value of p. Note that the obtained conditions are only sufficient for guaranteeing the negativity of (25). Theorem 2 is obtained by applying the Polya's approach to theorem 1 and by setting $p = 3$.

Theorem 2: ($p = 3$) Under the assumptions 1, given positive scalars σ and β, if there exists symmetric and positive definite matrices $\mathcal{X} \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n_f\times n_f}$ (with $n_f = n_u$) and matrices $M_i \in \mathbb{R}^{n_u\times n}$ and $N_i \in \mathbb{R}^{n\times n_y}$ and a positive scalar η solution to the optimization problem

$$\min \quad \eta \quad s.t. \tag{42}$$

$$\begin{pmatrix} \eta I & B_i^T P_2 - F_i C \\ (B_i^T P_2 - F_i C)^T & \eta I \end{pmatrix} > 0 \tag{43}$$

$$\begin{aligned}
&\mathcal{Q}_{ii} < 0, \quad i = 1, ..., r \\
&3\mathcal{Q}_{ii} + \mathcal{Q}_{ij} + \mathcal{Q}_{ji} < 0, \quad i, j = 1, ..., r, i \neq j \\
&3\mathcal{Q}_{ii} + \mathcal{Q}_{jj} + 3\mathcal{Q}_{ij} + 3\mathcal{Q}_{ji} < 0, \quad i, j = 1, ..., r, i \neq j \\
&6\mathcal{Q}_{ii} + 3\mathcal{Q}_{ij} + 3\mathcal{Q}_{ik} + 3\mathcal{Q}_{ji} + 3\mathcal{Q}_{ki} + \mathcal{Q}_{jk} + \mathcal{Q}_{kj} < 0 \\
&i, j, k = 1, ..., r, i < j < k \\
&3\mathcal{Q}_{ii} + 3\mathcal{Q}_{jj} + 6\mathcal{Q}_{ij} + 6\mathcal{Q}_{ji} + 3\mathcal{Q}_{ik} + 3\mathcal{Q}_{ki} \\
&+3\mathcal{Q}_{jk} + 3\mathcal{Q}_{kj} < 0, \\
&i, j, k = 1, ..., r, i < j < k
\end{aligned}$$

where $\mathcal{Q}_{ij}$ is defined in (12), then the state of the system $x(t)$, the state estimation error $e_x(t)$ and the fault estimation error $e_f(t)$ are bounded. The gains of the observer and the fault tolerant control are given by F_i, $L_i = P_2^{-1}N_i$ and $K_i = M_i\mathcal{X}^{-1}$.

VI. Simulation Example

To illustrate the performances of the proposed approach, let us consider the system (4) defined by the matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ 17.2941 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 3.5361 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 \\ -17.65 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ -17.63 \end{pmatrix}, \quad C = I_2$$

The weighting functions are given by $\mu_1(x(t)) = 1 - \frac{2}{\pi}|x_1(t)|$ and $\mu_2(x(t)) = 1 - \mu_1(x(t))$. Let us consider the fault $f(t)$ defined as follows

$$f(t) = \begin{cases} 0 & t < 20 \\ 1.4\sin(t) + 21 & 20 \leq t < 50 \\ 7.5\sin(2t) + 7.5 & 50 \leq t < 70 \\ -0.88u(t) & 70 \leq t \leq 100 \end{cases} \tag{44}$$

For $t \geq 70$ s, the fault $f(t)$ describes a loss of effectiveness of the actuator, which satisfies assumption **A3**. The first simulation is obtained by synthesizing a classical controller, without taking into account the faults, in the form $u(t) = -\sum_{i=1}^{r} \mu_i(\xi(t))K_i x(t)$ by using an approach proposed in [19]. For example, the gains K_i can be obtained by $K_i = M_i P^{-1}$ where P and M_i are solution of the LMIs

$$PA_i^T + PA_i - B_i M_j - M_j^T B_i^T < 0, \quad i, j = 1, 2 \tag{45}$$

With this control law, as shown in the figure 1, the states of the system converge to zero in fault free case, but in the faulty case the system performances are degraded from $t = 20$ s to $t = 70$ s and the system becomes unstable for $t \geq 70$ s. The proposed fault tolerant control is designed by

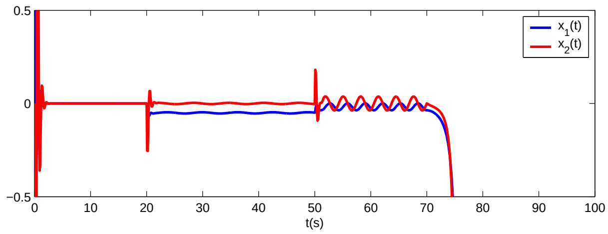

Fig. 1. System states with classical control

solving the optimization problem of theorem 1. For that, the parameter values $\sigma = 0.8$, $\Gamma = 44$ and $\beta = 10$ are chosen. The obtained gains of the observer and the controller are

$$L_1 = \begin{pmatrix} 0.52 & 1.22 \\ 17.24 & 0.27 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0.52 & 1.21 \\ 3.48 & 0.26 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 3.63 & -43.14 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 3.62 & -43.09 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 161.81 & -66.04 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 156.06 & -65.28 \end{pmatrix}$$

The figure 2 illustrates the results of the proposed control law obtained after solving the optimization problem of theorem 1. One can note that, with the fault $f(t)$ defined in (44), the performances are better than those of the classical control and the system remains stable for $t \geq 70$ (figure 2 (top)). The observer rapidly and accurately estimates the fault as shown in the figure 2 (bottom). In this example, the classical control cannot preserve the stability of the system when $\gamma \geq 0.88$ however (based on simulations not displayed here due to space limitation) it can be claimed that the proposed FTC strategy can tolerate faults until $\gamma = 0.97$.

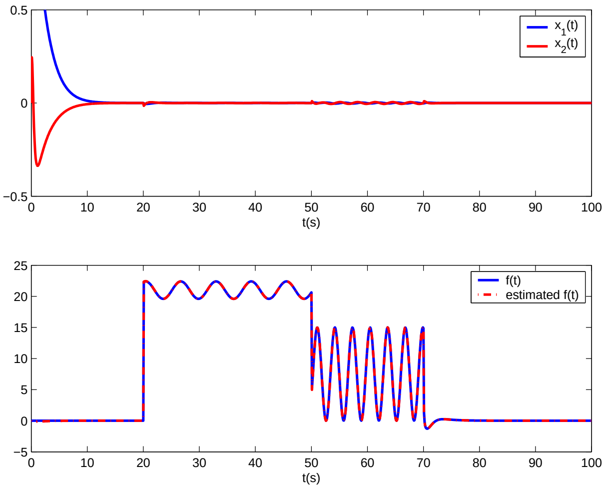

Fig. 2. Fault tolerant control : states of the system (top) - fault and its estimation (bottom)

In addition, this approach provides a rapid and accurate estimation of occurred actuator faults with the adaptive observer (figure 2 (bottom)) which constitutes a FDI block for diagnosis. If $f(t) = 7.5sin(2t)$, its derivative over the time is bounded by 15, then in this simulation example, the term $\delta = \frac{1}{\sigma} f_{1max}^2 \lambda_{max}(\Gamma^{-1} G^{-1} \Gamma^{-1}) = 0.0288$, and the term ε can be minimized by an appropriate choice of Γ to reduce the radius of the ball in which converge the estimation errors and then obtain a more accurate fault estimation.

VII. CONCLUSIONS AND FUTURE WORKS

This paper is dedicated to the study of a new actuator fault tolerant control for nonlinear systems in Takagi-Sugeno's form. The active fault tolerant control requires the simultaneous estimations of the state and fault, obtained by the proposed adaptive observer. This observer is able to estimate time varying faults with a good accuracy simultaneously with the estimation of the state. Furthermore, it gives the estimations rapidly which is important to preserve the performances of the system. The stability analysis is done with Lyapunov theory and ISS (Input-to-State Stability) is proved in the case of time varying faults, and asymptotic stability in the case of constant faults. Sufficient stability conditions are given in terms of LMI. In order to reduce the conservatism of the given conditions, Polya's theorem is used which allows to derive relaxed conditions for FTC design for nonlinear systems. Future works will concern the FTC of systems affected by both sensor and actuator fault and/or uncertainties and/or perturbations. It will also be interesting to study the case when a set of actuators is completely out of order, in this situation the dimensions of the matrices B_i and of the control vector $u(t)$ are decreased.

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