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# Brief Paper

# State estimation of Takagi–Sugeno systems with unmeasurable premise variables

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**Abstract:** This study is dedicated to the design of observers for non-linear systems described by Takagi–Sugeno (T–S) multiple models with unmeasurable premise variables. Furthermore, this T–S structure can represent a larger class of non-linear systems compared to the T–S systems with measurable premise variables. Considering the state of the system as a premise variable allows one to exactly represent the non-linear systems described by the general form $\dot{x} = f(x, u)$. Unfortunately, the developed methods for estimating the state of T–S systems with measured premise variable are not directly applicable for the systems that use the state as a premise variable. In the present paper, firstly, the design of observers for T–S systems with unmeasurable premise variable is proposed and sufficient convergence conditions are established by Lyapunov stability analysis. The linear matrix inequality (LMI) formalism is used in order to express the convergence conditions of the state estimation error in terms of LMI and to obtain the gains of the observer. Secondly, the proposed method is extended in order to attenuate energy-bounded unknown inputs such as disturbances. An academic example is proposed to compare some existing methods and the proposed one.

## 1 Introduction

Recently, the control, the observation and the diagnosis of non-linear systems received an important consideration. The incessant demand in terms of reliability and performance of the estimation and/or control has led to the use of non-linear models that are able to represent the system in a large domain of functioning. Therefore the obtained models may be very complex and the task of designing control laws and diagnostic algorithms become more difficult to achieve.

In recent years, the Takagi–Sugeno (T–S) structure, introduced in [1], provides a good way to represent non-linear systems thanks to its reduced mathematical complexity and its capabilities to describe a large class of non-linear systems. Indeed, T–S systems can approximate general non-linear systems or represent many of them [2] (Chapters 2 and 14). The T–S multiple model structure is based on the decomposition of the operating space of the system in several zones (e.g. neighbourhood of given operating points) and the behaviour of the system in each zone is represented by a linear local model. The contribution of each local model is quantified by the weighting functions. Thanks to these non-linear weighting functions, the overall behaviour of the system is represented by the bending of the local models. Furthermore, T–S formalism has the advantage that it allows one to use a part of the rich control system theory that had been developed for linear systems in the past years.

The paper is organised as follows: notations used in the paper and the presentation of the T–S multiple model structure are given in Section 2. Section 3 gives some background results on the control and observation of T–S non-linear systems in general, and presents some results on the problem of state estimation for T–S systems with unmeasurable premise variables. The main results are presented in the Sections 4 and 5, in the first one, a method to design observers is proposed using Lyapunov theory and, in the second one, the method is extended to

T−S systems with disturbances affecting the state and the output equations in order to reduce their influence on the state estimation. Finally, a numerical example is proposed in order to discuss the performances and limitations of the proposed observer, and to compare its results with those of the existing methods.

## 2 T–S structure and motivations

Let consider a non-linear system described by

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

The system (1) can be approximated or represented (according to the number $r$ of sub-models, chap 2 [2]) by the T−S structure

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \quad (2)$$

where $A_i \in R^{n\times n}$, $B_i \in \mathbb{R}^{n\times n_u}$, $C \in R^{n_y\times n} \cdot x(t) \in R^n$ is the state, $y(t) \in R^{n_y}$ represents the output of the system and $u(t) \in R^{n_u}$ is the input. The weighting functions satisfy the convex sum property expressed in the following equations

$$\begin{cases} 0 \le \mu_i(\xi(t)) \le 1 \\ \sum_{i=1}^{r} \mu_i(\xi(t)) = 1 \end{cases} \quad (3)$$

The weighing functions $\mu_i$ are generally non-linear and depend on the so-called decision variable $\xi(t)$ that can be external or internal to the system. When the decision variable is internal to the system, it can be measurable such as $\{u(t), y(t)\}$ or not measurable as the state $x(t)$ of the system.

The structure of the multiple model is simple and is considered as a universal approximator since it can represent any non-linear behaviour according to an adequate number $r$ of the local models. Thus $\dot{x}(t)$ appears to be a weighted sum of the contributions of each submodel $(A_i, B_i)$ state vector $\dot{x}_{(1)}(t), \ldots, \dot{x}_{(r)}(t)$ (Fig. 1).

The multiple model structure provides a mean to generalise the tools developed for linear systems to non-linear systems because of the structure (2) and properties (3).

In order to obtain a T−S system (2) from (1), different methods exist such as linearisation around some operating points, and using adequate weighting functions, the system (1) can be approximated. The most interesting and important way to obtain a T−S model is the well-known transformation by non-linear sector [2, 3]. Indeed, this transformation allows one to obtain an exact T−S representation of (1). It is proved in [4] that if the output of the system is chosen as a premise variable $\xi(t)$ and if this output is affected by disturbances (which is the case in all practical situations) the obtained T−S system does not represent precisely system (1). It is also pointed out that if the output is non-linear with respect to the state of the system it is difficult or impossible to obtain a T−S model by non-linear sector transformation with the output as a premise variable. That is why taking the state of the system as premise variable allows one to describe a wider class of non-linear systems.

In the context of diagnosis of T−S systems, the problem of fault isolating is difficult with only one model. Indeed, if the actuator fault isolation problem is considered, constructing a bank of observers in order to isolate faults is not possible: if a possibly faulty input is used as a premise variable, the estimation will not be decoupled from this input, even if it is considered as an unknown input. The same problem is encountered when trying to isolate the sensor faults with a T−S model using the output as a premise variable. The solution to this problem, which is largely used in the literature, is to develop two different T−S models for the same non-linear system. The first T−S model uses the input of the system as premise variable in order to isolate sensor faults, that is, faults affecting the system output. The second one uses the output of the system as premise variable to isolate actuator faults, that is, faults affecting the system input. The proposed solution is to develop only one T−S model

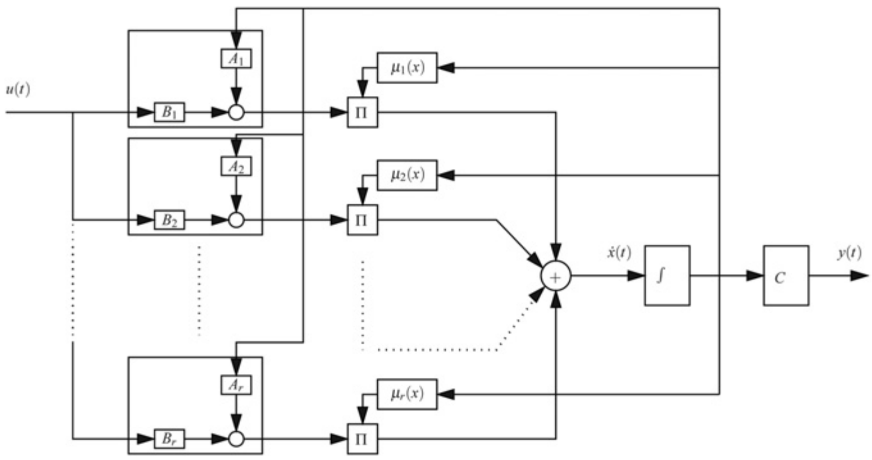


**Figure 1** *Evolution of the state in a multiple model with a common state vector*



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that uses the state of the system as premise variable for the non-linear system. This allows one to solve both the problems of isolating actuator and sensor faults. Hence, the problem of designing observers for this class of systems is very interesting.

Another important domain where the multiple model structure is used is the cryptanalysis and chaotic systems. In [5], based on this model by considering that the output of the system is the premise variable, a new observer design method is proposed in order to achieve synchronisation, but it is pointed out that using the unknown state as a premise variable will improve the synchronisation process security.

As a conclusion, the T–S multiple model with unmeasurable premise variables is an interesting structure because

- it can represent exactly the behaviour of the general non-linear system described by (1);
- a wider class of non-linear systems can be described by this structure compared to the TS models that use only measurable premise variables [4];
- only one T–S model is sufficient to construct observer banks for actuator and sensor fault isolation;
- the synchronisation security is improved in the cryptanalysis domain [5].

It will be seen in the next section that observer design for this class of systems is not so widely studied.

## 3 Background results

Many efforts have been devoted, these last years, to the problems of stability analysis and stabilisability of T–S non-linear systems. In [6], stability and controller design are studied by using the Lyapunov theory, and sufficient conditions are given in linear matrix inequality (LMI) formulation. In [7, 8] polyquadratic Lyapunov functions are used, and relaxed stability conditions are proposed. For discrete time T–S non-linear systems, a new non-quadratic Lyapunov function is introduced in [9]. The interest of this non-quadratic approach is that it significantly reduces the conservatism. This approach considers the Lyapunov function variations between $m$ samples (i.e. $k$ and $k+m$ where $m>1$) instead of variations on one sample (i.e. $k$ and $k+1$). When $m$ increases, the stability conditions become less restrictive. However, in the cited works, the weighting functions are assumed to be known at each time. The problem of control design for the class of T–S systems, where the premise variable is only depending on the unmeasurable state of the system, is not largely studied in the literature; nevertheless, we can cite the observer-based controller designed in [7], by considering that the premise variable is completely unmeasurable but is estimated by the observer. In [10], the authors consider the case where the premise variable is partially measurable. Based on a quadratic Lyapunov function and LMI formalism, a design method is proposed. Another approach is proposed in [11], where $H_\infty$ output feedback is designed by formulating the system in an uncertain equivalent form. In this work, the uncertainties come from the assumption that the premise variable is the state of the system which is unmeasurable.

Concerning the state estimation and state feedback control of T–S non-linear systems, we can cite [7]. In [12–14] the authors proposed the design of an observer and an unknown input observer via LMI approach, this allows the state estimation and the diagnosis for non-linear systems using observer bank-based method to detect and isolate actuator and sensor faults. Another robust model-based approach to fault detection with T–S systems is proposed in [15] by minimising the effects of the disturbances and maximising the effects of the faults on the residual. In the context of observer design and diagnosis for T–S non-linear systems with unmeasurable premise variables, few works have been published. The only work that may be encountered in the literature are the fuzzy Thau–Luenberger observer proposed in [16], which is an extension of the well-known Thau–Luenberger to T–S systems; the main result in this paper is the LMI formulation of conditions given to design the gains of the observer. The case of uncertain systems has also been considered and studied using sliding mode observers; applying the same idea, an unknown input observer is proposed in [17] for diagnosis. In [4], the $H_\infty$ filtering problem was dealt with in order to attenuate the effects of the disturbances on the state estimation error.

From the above background, we remark that there are only few works dealing with the observer design for non-linear systems described by T–S systems with unmeasurable premise variables. The method proposed in [16] is very conservative with regard to the considered Lipschitz constant. Indeed, in [18], the authors pointed out, on the inverted pendulum system, that even if the simulations are correct for the maximum admissible Lipschitz constant, this constant, which is obtained by maximisation, is far from the real Lipschitz constant of the system. The unknown input observer proposed in [17] suffers from the same problem and the proposed unknown input decoupling conditions are very conservative since the number of local models increases. The study given in [4] concerns only the $H_\infty$ filter design for T–S systems without input (i.e. $u(t)=0$).

In this paper, we propose a method to design observers for T–S non-linear systems with system state as premise variable. Based on the Laypunov theory, sufficient conditions are given for the stability of the state estimation error. The problem of observer design is reduced to the determination of a matrix gain by solving a set of LMIs obtained from the stability analysis. An extension of the method for the systems subject to norm-bounded exogenous inputs is presented. The gain of the observer is obtained by solving a feasibility problem under LMI constraints that ensure both the stability of the state estimation error and the minimisation of the effect of the exogenous inputs.

## 4 Observer design

In this section, the problem of designing observers for nonlinear systems described by (2) with unmeasurable premise variable (i.e. $\xi(t) = x(t)$) is considered. Let us consider the following matrices $A_0$, $\overline{A}_i$, $B_0$ and $\overline{B}_i$ defined by

$$A_0 = \frac{1}{r}\sum_{i=1}^{r} A_i \tag{4}$$

$$B_0 = \frac{1}{r}\sum_{i=1}^{r} B_i \tag{5}$$

and

$$\overline{A}_i = A_i - A_0 \tag{6}$$

$$\overline{B}_i = B_i - B_0 \tag{7}$$

The matrices $A_0$ and $B_0$ represent the mean of the set of matrices of the multiple model, which can be considered as the matrices that characterise a nominal model. The matrices $\overline{A}_i$ and $\overline{B}_i$ describe the variations around the nominal matrices. Substituting (4)–(7) into (2), the following equivalent system is obtained

$$\begin{cases} \dot{x}(t) = A_0 x(t) + B_0 u(t) + \sum_{i=1}^{r} \mu_i(x)(\overline{A}_i x(t) + \overline{B}_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{8}$$

which points out the role of the centres of the polytopes described by the vertices $A_i$ and $B_i$.

Based on the structure of the multiple-model (8), the following proportional observer is proposed

$$\begin{cases} \dot{\hat{x}}(t) = A_0 \hat{x}(t) + B_0 u(t) + \sum_{i=1}^{r} \mu_i(\hat{x})(\overline{A}_i \hat{x}(t) + \overline{B}_i u(t)) \\ \qquad + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{9}$$

Let the state estimation error be defined by

$$e(t) = x(t) - \hat{x}(t) \tag{10}$$

Using the equations of the system (8) and the observer (9), the dynamic of the state estimation error is given by

$$\dot{e}(t) = (A_0 - LC)e(t) + \sum_{i=1}^{r} (\overline{A}_i \delta_i(t) + \overline{B}_i \Delta_i) \tag{11}$$

where the perturbation terms are detailed below

$$\delta_i(t) = \mu_i(x(t))x(t) - \mu_i(\hat{x}(t))\hat{x}(t) \tag{12}$$

and

$$\Delta_i(t) = (\mu_i(x(t)) - \mu_i(\hat{x}(t)))u(t) \tag{13}$$

which unfortunately defined on the states of the system and its states observer.

*Assumption 1:* Throughout this paper, the following conditions hold

- The pair $(A_0, C)$ is observable.
- The input $u(t)$ is bounded $\|u_i(t)\| < \rho$, $\forall t$
- $|\mu_i(x)x - \mu_i(\hat{x})\hat{x}| \leq \alpha_i |x - \hat{x}|$
- $|(\mu_i(x) - \mu_i(\hat{x}))u| \leq \beta_i |x - \hat{x}|$

where $\alpha_i$ and $\beta_i$ are real matrices with all components being positive definite. The computation of the matrices $\alpha_i$ and $\beta_i$ is given in Appendix.

*Theorem 1 (Under Assumption 1):* Given the matrices $\alpha_i$, $\beta_i$ $i = 1, \ldots, r$, the state estimation error between the multiple model (8) and the observer (9) converges asymptotically to zero if there exists a positive-definite and symmetric matrix $P \in \mathbb{R}^{n\times n}$, a matrix $K \in \mathbb{R}^{n\times n_y}$ and symmetric positive-definite matrices $\Lambda_{1i} \in R^{n\times n}$ and $\Lambda_{2i} \in R^{n_u \times n}$ $(i = 1, \ldots, r)$ such that the following LMI holds

$$\begin{bmatrix} A_0^T P - C^T K^T + PA_0 - KC & \Xi_A & \Xi_B & \Xi_\alpha \Lambda_A & \Xi_\beta \Lambda_B \\ \Xi_A^T & -\Lambda_A & 0 & 0 & 0 \\ \Xi_B^T & 0 & -\Lambda_B & 0 & 0 \\ \Lambda_A^T \Xi_\alpha^T & 0 & 0 & -\Lambda_A & 0 \\ \Lambda_B^T \Xi_\beta^T & 0 & 0 & 0 & -\Lambda_B \end{bmatrix} < 0 \tag{14}$$

where

$$\Xi_A = [P\overline{A}_1 \ \cdots \ P\overline{A}_r] \tag{15}$$

$$\Xi_B = [P\overline{B}_1 \ \cdots \ P\overline{B}_r] \tag{16}$$

$$\Xi_\alpha = [\alpha_1 \ \cdots \ \alpha_r] \tag{17}$$

$$\Xi_\beta = [\beta_1 \ \cdots \ \beta_r] \tag{18}$$

$$\Lambda_A = \mathrm{diag}([\Lambda_{11} \ \cdots \ \Lambda_{1r}]) \tag{19}$$

$$\Lambda_B = \mathrm{diag}([\Lambda_{21} \ \cdots \ \Lambda_{2r}]) \tag{20}$$



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The gain of the observer is derived from

$$L = P^{-1}K \tag{21}$$

*Proof:* Consider the quadratic Lyapunov candidate function

$$V(e(t)) = e(t)^{\mathrm{T}}Pe(t), \quad P = P^{\mathrm{T}} > 0 \tag{22}$$

Its derivative with respect to time $t$ is given by

$$\dot{V}(e(t)) = \dot{e}(t)^{\mathrm{T}}Pe(t) + e(t)^{\mathrm{T}}P\dot{e}(t) \tag{23}$$

Then, using (11) the following is obtained

$$\begin{aligned}\dot{V}(e(t)) &= e(t)^{\mathrm{T}}((A_0 - LC)^{\mathrm{T}}P + P(A_0 - LC))e(t) \\ &\quad + 2e(t)^{\mathrm{T}}P\sum_{i=1}^{r}\overline{A}_i\delta_i(t) + 2e(t)^{\mathrm{T}}P\sum_{i=1}^{r}\overline{B}_i\Delta_i(t)\end{aligned} \tag{24}$$

*Lemma 1:* For any matrices $X$ and $Y$ with appropriate dimensions, and any positive-definite matrix $\Lambda$, the following property holds

$$X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \le X^{\mathrm{T}}\Lambda X + Y^{\mathrm{T}}\Lambda^{-1}Y, \quad \Lambda > 0 \tag{25}$$

By applying Lemma 1 and Assumption 1, we obtain

$$\begin{aligned}&\delta_i(t)^{\mathrm{T}}\overline{A}_i^{\mathrm{T}}Pe(t) + e(t)^{\mathrm{T}}P\overline{A}_i\delta_i(t) \\ &\quad \le \delta_i(t)^{\mathrm{T}}\Lambda_{1i}\delta_i(t) + e(t)^{\mathrm{T}}P\overline{A}_i\Lambda_{1i}^{-1}\overline{A}_i^{\mathrm{T}}Pe(t) \\ &\quad \le e(t)^{\mathrm{T}}\alpha_i^{\mathrm{T}}\Lambda_{1i}\alpha_i e(t) + e(t)^{\mathrm{T}}P\overline{A}_i\Lambda_{1i}^{-1}\overline{A}_i^{\mathrm{T}}Pe(t)\end{aligned} \tag{26}$$

$$\begin{aligned}&\Delta_i(t)^{\mathrm{T}}\overline{B}_i^{\mathrm{T}}Pe(t) + e(t)^{\mathrm{T}}P\overline{B}_i\Delta_i(t) \\ &\quad \le \Delta_i(t)^{\mathrm{T}}\Lambda_{2i}\Delta_i(t) + e(t)^{\mathrm{T}}P\overline{B}_i\Lambda_{2i}^{-1}\overline{B}_i^{\mathrm{T}}Pe(t) \\ &\quad \le e(t)^{\mathrm{T}}\beta_i^{\mathrm{T}}\Lambda_{2i}\beta_i e(t) + e(t)^{\mathrm{T}}P\overline{B}_i\Lambda_{2i}^{-1}\overline{B}_i^{\mathrm{T}}Pe(t)\end{aligned} \tag{27}$$

Taking into account (26) and (27), the derivative of the Lyapunov function (24) can be, then, bounded as follows

$$\begin{aligned}\dot{V}(e(t)) &\le e(t)^{\mathrm{T}}\Big((A_0 - LC)^{\mathrm{T}}P + P(A_0 - LC) \\ &\quad + \sum_{i=1}^{r}(P\overline{A}_i\Lambda_{1i}^{-1}\overline{A}_i^{\mathrm{T}}P + P\overline{B}_i\Lambda_{2i}^{-1}\overline{B}_i^{\mathrm{T}}P \\ &\quad + \alpha_i^{\mathrm{T}}\Lambda_{1i}\alpha_i + \beta_i^{\mathrm{T}}\Lambda_{2i}\beta_i)\Big)e(t)\end{aligned} \tag{28}$$

The negativity of the derivative of the Lyapunov function (28) is guaranteed if

$$\begin{aligned}&(A_0 - LC)^{\mathrm{T}}P + P(A_0 - LC) + \sum_{i=1}^{r}(P\overline{A}_i\Lambda_{1i}^{-1}\overline{A}_i^{\mathrm{T}}P \\ &\quad + P\overline{B}_i\Lambda_{2i}^{-1}\overline{B}_i^{\mathrm{T}}P + \alpha_i^{\mathrm{T}}\Lambda_{1i}\alpha_i + \beta_i^{\mathrm{T}}\Lambda_{2i}\beta_i) < 0\end{aligned} \tag{29}$$

which can be written, using definitions (15)–(20), as follows

$$\begin{aligned}&(A_0 - LC)^{\mathrm{T}}P + P(A_0 - LC) + \Xi_A\Lambda_A^{-1}\Xi_A^{\mathrm{T}} + \Xi_B\Lambda_B^{-1}\Xi_B^{\mathrm{T}} \\ &\quad + \Xi_\alpha\Lambda_A\Xi_\alpha^{\mathrm{T}} + \Xi_\beta\Lambda_B\Xi_\beta^{\mathrm{T}} < 0\end{aligned} \tag{30}$$

with the Schur complement and the variable change $K = PL$, (30) is equivalent to (14). □

## 5 $\mathcal{L}_2$ attenuating observer

Let us consider the more general situation where T–S systems are subject to disturbances $\omega(t)$

$$\begin{cases}\dot{x}(t) = \sum_{i=1}^{r}\mu_i(x)(A_ix(t) + B_iu(t) + E_i\omega(t)) \\ y(t) = Cx(t) + G\omega(t)\end{cases} \tag{31}$$

where $E_i \in R^{n\times n_\omega}$ and $G \in R^{n_y\times n_\omega}$ are constant matrices giving the influence of the disturbances, respectively, on the state and output equations. It is assumed that $\omega(t)$ is an energy-bounded disturbance vector, that is, $\omega(t) \in \mathcal{L}_2$. Note that the situation where the disturbances affecting the dynamic parts and the static parts of the system are decoupled may be adressed by using an appropriate definition of matrices $E_i$ and $G$.

The objective is to adapt the design of the observer (9) in order to estimate the state of the system (31) guaranteeing disturbance attenuation at a certain level.

Transformations made in the previous section by introducing matrices $A_0$ and $B_0$ are used to obtain the following equivalent system of the system (31)

$$\begin{cases}\dot{x}(t) = A_0x(t) + B_0u(t) + \sum_{i=1}^{r}\mu_i(x)(\overline{A}_ix(t) + \overline{B}_iu(t) + E_i\omega(t)) \\ y(t) = Cx(t) + G\omega(t)\end{cases} \tag{32}$$

The dynamic of the state estimation error $e(t) = x(t) - \hat{x}(t)$ between the system (32) and the observer (9) is now the solution of the differential equation

$$\begin{aligned}\dot{e}(t) &= (A_0 - LC)e(t) + \sum_{i=1}^{r}(\overline{A}_i\delta_i(t) + \overline{B}_i\Delta_i(t) \\ &\quad + \mu_i(x)(E_i - LG)\omega(t))\end{aligned} \tag{33}$$

where the perturbation terms $\delta_i(t)$ and $\Delta_i(t)$ are defined by (12) and (13), respectively.

The problem of robust state estimation error is reduced to the determination of the observer gain $L$ ensuring an asymptotic convergence of $e(t)$ towards zero if $\omega(t) = 0$ (34) and simultanuously ensuring a bounded ratio of the energy of the disturbance signal and state estimation error (35) when $\omega(t) \neq 0$, that is

$$\lim_{t\to\infty} e(t) = 0 \quad \text{for } \omega(t) = 0 \tag{34}$$

$$\|e(t)\|_M \leq \gamma \|\omega(t)\|_N \quad \text{for} \quad \omega(t) \neq 0 \quad \text{and} \quad e(0) = 0 \tag{35}$$

where

$$\|e(t)\|_M = \sqrt{\int_0^\infty e(t)^{\mathrm{T}} M e(t)\, \mathrm{d}t},$$
$$\|\omega(t)\|_N = \sqrt{\int_0^\infty \omega(t)^{\mathrm{T}} N \omega(t)\, \mathrm{d}t} \tag{36}$$

$M$ and $N$ are known weighting matrices and $\gamma > 0$ is the attenuation level of the disturbances. It is well known [19] that to satisfy the constraints (34) and (35), it is sufficient to find a Lyapunov function $V$ such that

$$\dot{V}(t) + e(t)^{\mathrm{T}} M e(t) - \gamma^2 \omega(t)^{\mathrm{T}} N \omega(t) < 0 \tag{37}$$

To justify this proposition, let us remark that, on the one hand, if $\omega(t) = 0$, (37) implies that $\dot{V} < 0$, thus from the Lyapunov theory, an asymptotic convergence of the state estimation error $e(t)$ is obtained, then we have (34). On the other hand, if $\omega(t) \neq 0$, (37) implies

$$V(t) + \int_0^t e(t)^{\mathrm{T}} M e(t)\, \mathrm{d}t - \gamma^2 \int_0^t \omega(t)^{\mathrm{T}} N \omega(t)\, \mathrm{d}t < 0 \tag{38}$$

By definition, the Lyapunov function satisfies $V(t) > 0$, then

$$\|e(t)\|_2 \leq \gamma \|\omega(t)\|_2 \tag{39}$$

which corresponds to (35).

Sufficient conditions to synthesise a robust observer in order to attenuate the influence of the disturbance $\omega(t)$ on the state estimation error are proposed in the following theorem.

*Theorem 2 (Under Assumption 1):* Given the attenuation level $\gamma > 0$, the matrices $M$, $N$, $\alpha_i$ and $\beta_i$ $i = 1, \ldots, r$, an observer (9) for the system (31) satisfying (34) and (35) exist if there exists a symmetric and positive matrix $P \in R^{n\times n}$, a matrix $K \in R^{n\times n_y}$ and symmetric definite positive matrices $\Lambda_{1i} \in R^{n\times n}$ and $\Lambda_{2i} \in R^{n_u\times n}$, $(i = 1, \ldots, r)$ such that the following LMIs are feasible (see (40))

where

$$\Xi_A = [\, P\bar{A}_1 \quad \cdots \quad P\bar{A}_r \,] \tag{41}$$

$$\Xi_B = [\, P\bar{B}_1 \quad \cdots \quad P\bar{B}_r \,] \tag{42}$$

$$\Xi_\alpha = [\, \alpha_1 \quad \cdots \quad \alpha_r \,] \tag{43}$$

$$\Xi_\beta = [\, \beta_1 \quad \cdots \quad \beta_r \,] \tag{44}$$

$$\Lambda_A = \mathrm{diag}([\, \Lambda_{11} \quad \cdots \quad \Lambda_{1r} \,]) \tag{45}$$

$$\Lambda_B = \mathrm{diag}([\, \Lambda_{21} \quad \cdots \quad \Lambda_{2r} \,]) \tag{46}$$

The gain $L$ of the observer is derived from

$$L = P^{-1} K \tag{47}$$

*Proof:* Let us consider the Lyapunov function (22). With (33), the derivative of the Lyapunov function is calculated in a similar way as in the proof of Theorem 1, and it is given by

$$\begin{aligned}\dot{V}(t) = {} & e(t)((A_0 - LC)^{\mathrm{T}} P + P(A_0 - LC) + \Xi_A \Lambda_A^{-1} \Xi_A^{\mathrm{T}} \\ & + \Xi_B \Lambda_B^{-1} \Xi_B^{\mathrm{T}} + \Xi_\alpha \Lambda_A \Xi_\alpha^{\mathrm{T}} + \Xi_\beta \Lambda_B \Xi_\beta^{\mathrm{T}}) e(t) \\ & + 2e(t)^{\mathrm{T}} P \sum_{i=1}^{r} \mu_i(x)(E_i - LG)\omega(t)\end{aligned} \tag{48}$$

The convex property of the sum of the weighting functions

$$\begin{bmatrix} A_0^{\mathrm{T}} P - C^{\mathrm{T}} K^{\mathrm{T}} + PA_0 - KC + M & \Xi_A & \Xi_B & \Xi_\alpha \Lambda_A & \Xi_\beta \Lambda_B & PE_i - KG \\ \Xi_A^{\mathrm{T}} & -\Lambda_A & 0 & 0 & 0 & 0 \\ \Xi_B^{\mathrm{T}} & 0 & -\Lambda_B & 0 & 0 & 0 \\ \Lambda_A^{\mathrm{T}} \Xi_\alpha^{\mathrm{T}} & 0 & 0 & -\Lambda_A & 0 & 0 \\ \Lambda_B^{\mathrm{T}} \Xi_\beta^{\mathrm{T}} & 0 & 0 & 0 & -\Lambda_B & 0 \\ E_i^{\mathrm{T}} P - G^{\mathrm{T}} K^{\mathrm{T}} & 0 & 0 & 0 & 0 & -\gamma^2 N \end{bmatrix} < 0, \quad i = 1, \ldots, r \tag{40}$$



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allows one to write

$$\sum_{i=1}^{r}\mu_i(x)(e(t)^{\mathrm{T}}((A_0-LC)^{\mathrm{T}}P+P(A_0-LC)+\Xi_A\Lambda_A^{-1}\Xi_A^{\mathrm{T}}+\Xi_B\Lambda_B^{-1}\Xi_B^{\mathrm{T}}+\Xi_\alpha\Lambda_A\Xi_\alpha^{\mathrm{T}}+\Xi_\beta\Lambda_B\Xi_\beta^{\mathrm{T}})e(t)+2e(t)^{\mathrm{T}}P(E_i-LG)\omega(t))<0 \quad (49)$$

Substituting (49) into (37), the following matrix inequality is obtained

$$\sum_{i=1}^{r}\mu_i(x)\zeta(t)^{\mathrm{T}}\begin{bmatrix}\Psi & P(E_i-LG)\\ (E_i-LG)^{\mathrm{T}}P & -\gamma^2 N\end{bmatrix}\zeta(t)<0 \quad (50)$$

where $\zeta(t)=[e(t)^{\mathrm{T}}\ \omega(t)^{\mathrm{T}}]^{\mathrm{T}}$ and

$$\Psi=(A_0-LC)^{\mathrm{T}}P+P(A_0-LC)+\Xi_A\Lambda_A^{-1}\Xi_A^{\mathrm{T}}+\Xi_B\Lambda_B^{-1}\Xi_B^{\mathrm{T}}+\Xi_\alpha\Lambda_A\Xi_\alpha^{\mathrm{T}}+\Xi_\beta\Lambda_B\Xi_\beta^{\mathrm{T}}+M \quad (51)$$

With (3), the inequality (50) is fullfilled if (see (52))

$$\begin{bmatrix}(A_0-LC)^{\mathrm{T}}P+P(A_0-LC)+\Xi_A\Lambda_A^{-1}\Xi_A^{\mathrm{T}}+\Xi_B\Lambda_B^{-1}\Xi_B^{\mathrm{T}}+\Xi_\alpha\Lambda_A\Xi_\alpha^{\mathrm{T}}+\Xi_\beta\Lambda_B\Xi_\beta^{\mathrm{T}}+M & P(E_i-LG)\\ (E_i-LG)^{\mathrm{T}}P & -\gamma^2N\end{bmatrix}<0 \quad (52)$$

$$i=1,\ldots,r$$

The LMIs given in Theorem 2 are obtained from (52) by using the Schur complement (see [19]) and the change of variable $K=PL$. □

*Remark 1:* Note that an optimally robust observer can be obtained by considering $\gamma$ as a variable to minimise. Since (52) is non-linear in $\gamma$, the change of variable $\eta=\gamma^2$ can be used to obtain an LMI in $\eta$. The optimally robust observer is obtained by solving the following problem

$$\min_{L,\eta}\ \eta \quad \text{subject to (40)} \quad (53)$$

The optimal attenuation level of the disturbances is obtained by

$$\gamma=\sqrt{\eta} \quad (54)$$

*Remark 2:* In order to improve the quality of the state estimation of the system, it is possible to develop a bank of observers for each state separately. This approach is based on the choice of the matrices $M$ and $N$ so as to minimise the disturbance influence on only one state for each observer. Therefore, for each observer the constraint of $\mathcal{L}_2$ norm minimisation is relaxed for all state variables but one. As a consequence, the attenuation levels will be less or equal to the one found in (54), and thus the state estimation is improved.

## 6 Pole assignment

In this section, consider the observer (9). Using the second method of Lyapunov, the convergence of the state estimation error to zero is guaranteed and the eigenvalues of $(A_0-LC)$ are assigned in a specific region $\mathbb{D}$.

*Definition 1:* The subset $\mathcal{D}$ of the complex left half plane is called LMI region if there exists a matrix $\alpha\in R^{n\times n}$ and matrix $\beta\in R^{n\times n}$ such as

$$\mathcal{D}=\{z\in C: f_D(z)=\alpha+\beta z+\beta^{\mathrm{T}}\bar{z}<0\} \quad (55)$$

The matrix $A$ is called $\mathcal{D}$-stable if all its eigenvalues are located in the domain $\mathcal{D}$ of the complex left half plane. The first idea is to place the eigenvalues in a vertical band ($S_1$) defined by

$$S_1=\{z/-\sigma_{\min}<\Re e(z)<-\sigma_{\max}<0\} \quad (56)$$

Another more interesting idea, since it limits the imaginary parts of the eigenvalues, is to place them in a region ($S_2$) defined by

$$S_2=\{z/|z|<R,\ \Re e(z)<-\sigma_{\max}<0\} \quad (57)$$

In order to assign the eigenvalues of $(A_0-LC)$ in the region $S_1$ to satisfy

$$\Re e(\lambda)<-\sigma_{\max}$$

it is sufficient that $(A_0-LC)+\sigma_{\max}I$ is stable, we obtain

$$(A_0-LC)^{\mathrm{T}}P+P(A_0-LC)+2\sigma_{\max}P<0 \quad (58)$$

A similar reasoning on the bound $\sigma_{\min}$ gives

$$-((A_0-LC)^{\mathrm{T}}P+P(A_0-LC)+2\sigma_{\min}P)<0 \quad (59)$$

Finally, in order to design the observer while assigning the eigenvalues of $(A_0-LC)$ in the region $S_1$, it is sufficient to solve the LMIs given in Theorem 1 with the conditions (58) and (59). The eigenvalues assignment in a circle of radius $R$ and centre (0.0) is given in [20] by

$$\begin{pmatrix}-RP & P(A_0-LC)\\ (A_0-LC)^{\mathrm{T}}P & -RP\end{pmatrix}<0,\quad P=P^{\mathrm{T}}>0 \quad (60)$$


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The design of the observer with eigenvalues assignement in the region $S_2$ is realised by solving the LMIs in Theorem 1 with the conditions (58) and (60). (For more details see [20].)

## 7 Simulation example

Consider a one-link manipulator with revolute joints actuated by a DC motor represented in Fig. 2 [21] whose model is defined by

$$\begin{cases} \dot{\theta}_m(t) = \omega_m(t) \\ \dot{\omega}_m(t) = \dfrac{k}{J_m}(\theta_l(t) - \theta_m(t)) - \dfrac{B}{J_m}\omega_m(t) + \dfrac{K_\tau}{J_m}u(t) \\ \dot{\theta}_l(t) = \omega_l(t) \\ \dot{\omega}_l(t) = -\dfrac{k}{J_l}(\theta_l(t) - \theta_m(t)) - \dfrac{mgh}{J_l}\sin(\theta_l(t)) \end{cases} \tag{61}$$

where $\theta_m(t)$ stands for the angular position of the motor, $\omega_m(t)$ is the angular velocity of the motor, $\theta_l(t)$ is the angular position of the link and $\omega_l(t)$ is the angular velocity of the link. The input signal is given by $u(t) = \sin(t)$, and the initial condition are $x_0 = [1\ 0\ 3\ 0]^T$ for the system and $\hat{x}_0 = [0.5\ 0.5\ 0.5\ 0.5]^T$ for the observer. The state representation is

$$\dot{x}(t) = Ax(t) + f(x(t)) + Bu(t) + E\omega(t),$$
$$y(t) = Cx(t) + G\omega(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} \theta_m \\ \omega_m \\ \theta_l \\ \omega_l \end{bmatrix}, \quad f(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33\sin(x_3) \end{bmatrix}, \quad E = \begin{bmatrix} 0.5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

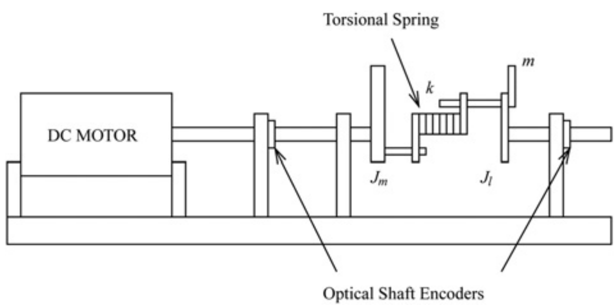


**Figure 2** *Flexible link joint robot*

By using a non-linear sector transformation approach [2], a multiple model representation of the above system, which describes exactly the behaviour of the original model, is given by (2) with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -22.83 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -18.77 & 0 \end{bmatrix}$$

$$B_1 = B_2 = B$$

$$\begin{cases} \mu_1(z(t)) = \dfrac{z(t) + 0.2172}{1.2172} \\ \mu_2(z(t)) = \dfrac{1 - z(t)}{1.2172} \end{cases} \tag{62}$$

where $z(t) = \sin(x_3)/x_3$. Fig. 3 presents the states of the original model and those of the multiple model and we can clearely seen that the multiple model represents exactely the original model of the system. The values taken by the weighting functions $\mu_i$ are illustrated in Fig. 4. The matrices $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are computed from the method given in Appendix and they are defined by

$$\alpha_1 = \begin{bmatrix} 1 & 0 & 9.7601 & 0 \\ 0 & 1 & 5.78 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3.6176 & 1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 1 & 0 & 6.1 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0.8216 & 0 \\ 0 & 0 & 3.6 & 1 \end{bmatrix},$$

$$\beta_1 = \beta_2 = \begin{bmatrix} 0 & 0 & 0.42 & 0 \end{bmatrix}$$

As a first simulation, we propose to compare the proposed method with the exsiting works. This comparison is based on the obtained state estimation errors, in the perturbation free case ($\omega(t) = 0$). Secondly, the case in the presence of perturbation is considered and a comparison between Theorems 1 and 2 is performed.

### *7.1 Comparison between the proposed and the existing approaches*

Solving the LMI constraint given in Theorem 1 simultaneously with the LMIs (58) and (60) in order to ensure the pole clustering in the region $S_2$ defined by




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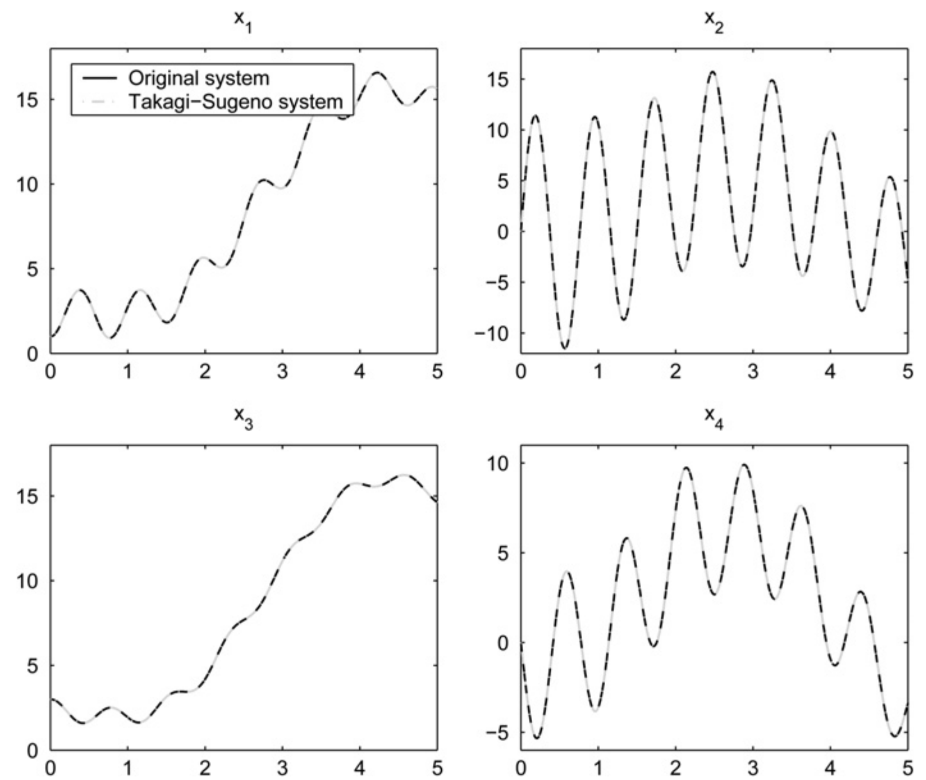


**Figure 3** *States of the original and multiple model*

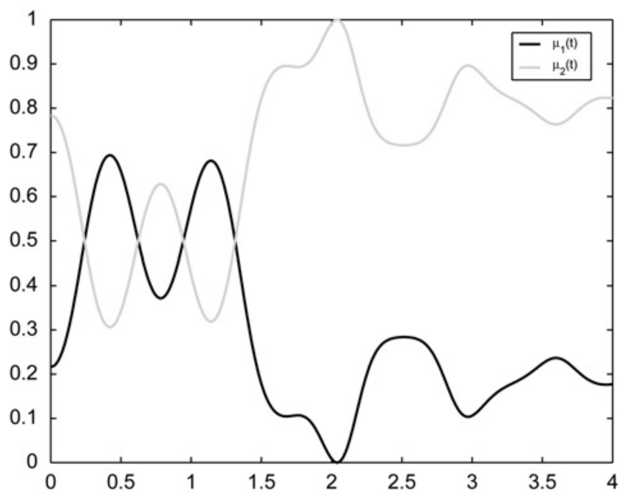

**Figure 4** *Weighting functions*

$\sigma_{max} = 9.5$ and $R = 10$, the following matrices are obtained

$$P = \begin{bmatrix} 119.1067 & -7.4369 & 29.0869 & -2.5598 \\ -7.4369 & 43.0543 & -175.1685 & 16.6264 \\ 29.0869 & -175.1685 & 730.7144 & -72.0210 \\ -2.5598 & 16.6264 & -72.0210 & 7.5085 \end{bmatrix}$$

$$L = \begin{bmatrix} 9.7307 & 1.0000 \\ -54.3794 & 27.5117 \\ -1.9395 & 5.4322 \\ 14.3097 & 7.6348 \end{bmatrix}$$

$$\Lambda_{11} = \begin{bmatrix} 195.75 & -170.54 & -59.42 & -57.15 \\ -170.54 & 160.15 & -24.78 & 47.25 \\ -59.42 & -24.78 & 854.61 & -8.62 \\ -57.15 & 47.25 & -8.62 & 22.85 \end{bmatrix},$$

$$\Lambda_{12} = \begin{bmatrix} -128.31 & 75.00 & -27.00 & 21.83 \\ -65.04 & -27.00 & 847.63 & -9.14 \\ -42.76 & 21.83 & -9.14 & 10.83 \end{bmatrix}$$

$$\Lambda_{21} = \Lambda_{22} = 7.05$$

The simulation results are depicted in Figs. 5 and 6. The proposed observer is compared with the observers given in [22, 23].

From this example, it is clear that the proposed approach gives better results compared to the methods proposed by Rajamani in [22] and Raghavan [23]. Furthermore, the method developed in [18], on the observer design for T–S systems with unmeasurable premise variables, cannot be used for this example because the proposed LMIs have no solution due to the value of the considerd Lipschitz constant.

## 7.2 Comparison between Theorems 1 and 2

The perturbation $\omega(t)$ is considered in this section in order to illustrate the attenuation problem given in Theorem 2. The considered perturbation is a zero mean random signal bounded by 0.5. The LMIs in Theorems 1 and 2 are

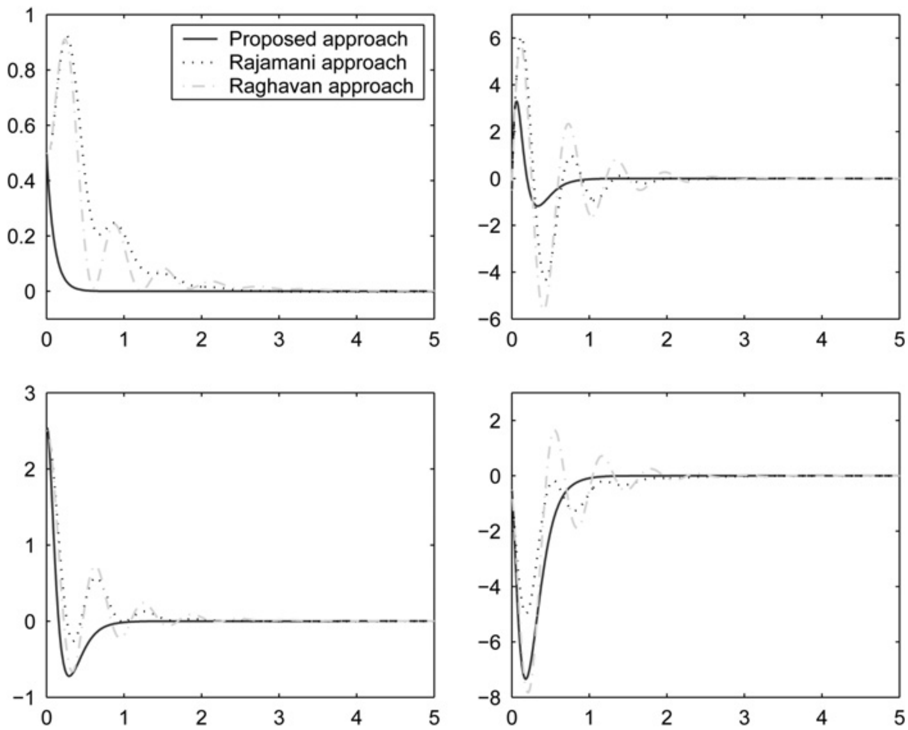


**Figure 5** *State estimation errors*

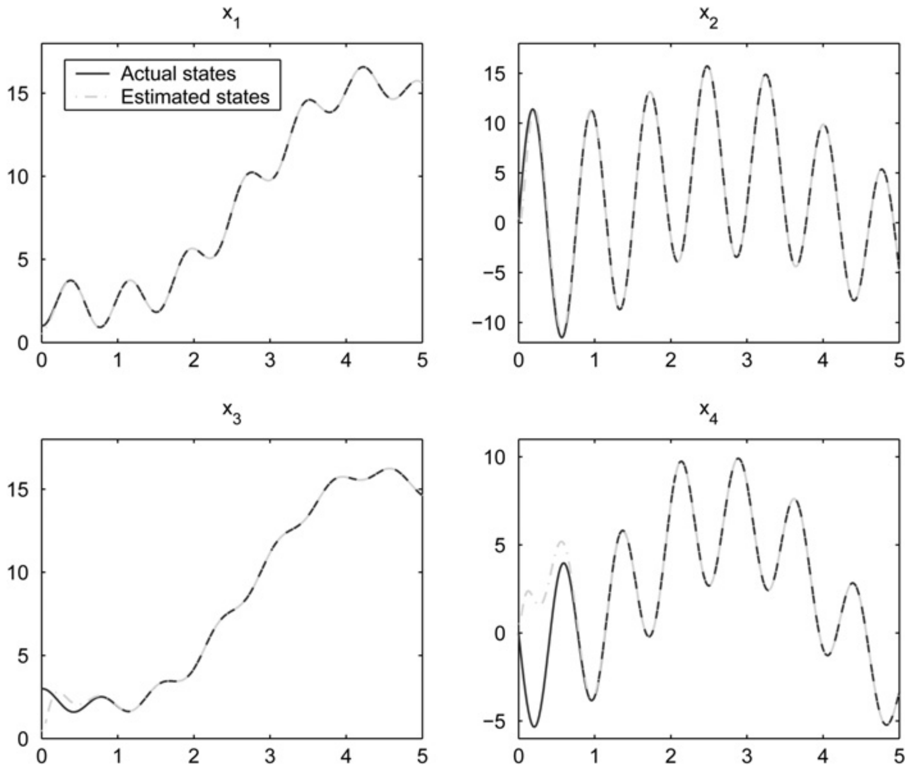


**Figure 6** *Actual states and their estimations*

solved with pole assignment in the region $S_2$ defined by the $\sigma_{max} = 1$ and $R = 10$. The matrix $M$ is the identity ($M = I_{4\times4}$) and $N = 1$. The obtained attenuation level is $\gamma = 2.82$. Fig. 7 shows the state estimation errors obtained with Theorems 1 and 2.

In Fig. 7, the state estimation errors are depicted. One can see that the results obtained by using Theorem 2 are better than those obtained using Theorem 1 where the perturbation is not taken into account. It is also possible to reduce the ascillatory phenomenon by changing the



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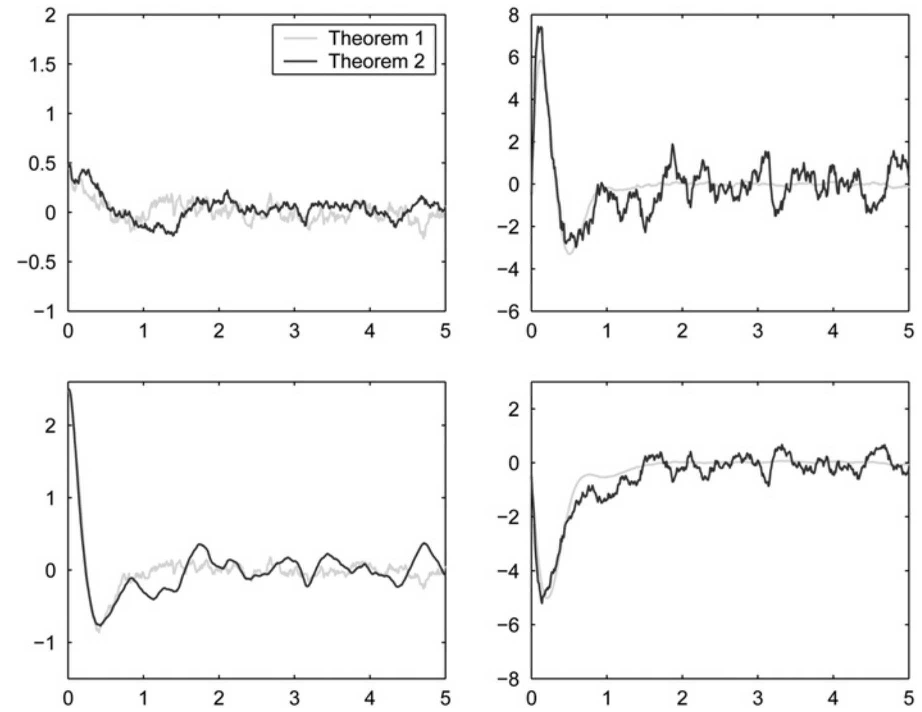


**Figure 7** *Comparison between the results obtained with Theorems 1 and 2*

parameters $\alpha$ and $R$ of the region in order to reduce the imaginary parts of the eigenvalues of the matrix $(A_0 - LC)$.

## 8 Conclusion

The problem of observer design for systems described by T–S multiple model with unmeasurable premise variables has been investigated. The convergence of the state estimation error is established by using the Lyapunov theory and a quadratic Lyapunov function candidate. The proposed method is extended to estimate the state of the system subject to energy-bounded exogenous disturbances. The convergence conditions are expressed in the LMIs formulation. For the future works, it will be interesting to study and reduce the conservatism of the LMIs given in Theorems 1 and 2 to propose relaxed conditions by using, for example, a non-quadratic Lyapunov candidate function.

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## 10 Appendix: Evaluation of the Lipschitz matrices

The calculation of $\alpha_i$ and $\beta_i$ introduced in Assumption 1 of Section 4 is given by the following steps: Consider a function $f(x): x \in R^n \to R^n$ defined as follows

$$f(x) = \left[ f_1(x)^{\mathrm{T}} \quad \cdots \quad f_n(x)^{\mathrm{T}} \right]^{\mathrm{T}}, \quad x = \left[ x_1^{\mathrm{T}} \quad \cdots \quad x_n^{\mathrm{T}} \right]^{\mathrm{T}}$$

The Taylor formula at order zero with an integral remainder term of $f(x)$ around $\hat{x}$ is

$$f_i(x) - f_i(\hat{x}) = \int_{\hat{x}_1}^{x_1} \frac{\partial f_i}{\partial x_1}(t)\,\mathrm{d}t + \cdots + \int_{\hat{x}_n}^{x_n} \frac{\partial f_i}{\partial x_n}(t)\,\mathrm{d}t, \quad i \in \{1, \ldots, n\} \tag{63}$$

Each function variation can be bounded as follows

$$|f_i(x) - f_i(\hat{x})| \le \int_{\hat{x}_1}^{x_1} \left| \frac{\partial f_1}{\partial x_1(t)} \right| \mathrm{d}t + \cdots + \int_{\hat{x}_n}^{x_n} \left| \frac{\partial f_1}{\partial x_n}(t) \right| \mathrm{d}t \tag{64}$$

Let define $a_{ij} = \max_{t \in [x_j\, \hat{x}_j]} |(\partial f_i / \partial x_j(t)|\ i, j \in \{1, \ldots, n\}$. The interval $[x_j\ \hat{x}_j]$ is not known, so $a_{ij}$ is calculated for $t \in R$. We obtain $a_{ij} = \max_{t \in R} |(\partial f_i / \partial x_j)(t)|$. Then, (64) can be rewritten as follows

$$|f_i(x) - f_i(\hat{x})| \le a_{i1}|x_1 - \hat{x}_1| + \cdots + a_{in}|x_n - \hat{x}_n|, \\ i \in \{1, \ldots, n\}$$

By rewriting the above inequalities in matrix form, we obtain

$$|f(x) - f(\hat{x})| \le J|x - \hat{x}| \tag{65}$$

where

$$J = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Then, considering Assumption 1, inequality (65) may be applied successively for $f(x) = \mu_i(x)x$ and $f(x) = \mu_i(x)u$ in order to obtain the matrices $\alpha_i$ and $\beta_i$. If the weighting functions are Lipschitz, the input of the system is bounded, then the state is also bounded and the parameters $a_{ij}$ exist.



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