On the simultaneous state and unknown inputs estimation of complex systems via a multiple model strategy

Rodolfo Orjuela, Benoît Marx, José Ragot and Didier Maquin
Centre de Recherche en Automatique de Nancy (CRAN)
Nancy-Université, CNRS
2 avenue de la Forêt de Haye F-54516 Vandoeuvre-lès-Nancy
rodolfo.orjuela, benoit.marx, jose.ragot, didier.maquin@ensem.inpl-nancy.fr

Abstract

This paper addresses the analysis and design of unknown input observer in order to provide both state and unknown input estimation of complex systems modelled with the help of a particular class of multiple model. The proposed observer uses the multi-integral strategy successfully employed in the classic linear control theory and known for its robustness properties. The observer design is based on the representation of the system via a multiple model, known as *decoupled multiple model*. This structure of multiple model allows to use submodels with different number of states and this fact constitutes the main advantage of the proposed observer with respect to the classic multiple model structure where the submodels have the same dimension. It is shown how the gains of the suggested observer can be obtained by solving a LMI optimal problem. An academic example is also proposed in order to illustrate the proposed methodology.

Keywords. *nonlinear systems, multiple models, state estimation, unknown input multi-integral observer,*

1 Introduction

In many real-world applications the direct measurement of the state variables of a system can be very difficult to obtain, or even impossible, due for example to physical constraints and/or economical restrictions. The use of an estimation of the state, using an observer, instead of its measured value, provided by sensors, is a solution largely adopted in order to avoid these problems. However, it is well known that in many situations some inputs of the system are inaccessible or considered for simplicity as such. These unmeasurable signals, denoted as unknown inputs (UI), have a serious impact on the state reconstruction and can be at the origin of biased estimations if they are not correctly taken into consideration.

Hence, both state and UI estimations have been intensively investigated for years because these estimations are of great use in several engineering applications such as control, supervision and fault-tolerant control. Indeed, an actuator/sensor failure or an abnormal behaviour of an internal component of the system can be regarded as unmeasured signals modelled by UIs. Hence, the UI estimations can be considered as fault symptoms of the system for fault detection and isolation in order to improve the safety of the system.

The so-called unknown input observer (UIO) is usually employed as a mean to provide both state and UI estimations of a system. There are several approaches for unknown input observer design for system represented by linear time-invariant models. In a general way, the UIO design consists in finding a projection operator or a matrix transformation fulfilling a set of algebraic equations in order to decouple the UI and the estimation error (see [1, 2, 3, 4, 5] and references therein). Note however that, as pointed out in [6], output derivatives are required in order to perform an UI estimation. Undoubtedly the sensor noises are unavoidable and derivative of such signals may be unsuitable.

Another attractive way to obtain a simultaneous estimation of the states and the UI is to use the Proportional-Integral observer (PIO). This observer known for its robustness properties [7, 8] has been successfully employed in order to cope with the state and the UI estimations when the UI is characterised by low frequency signals (constant or slowly varying signals) [9, 10, 11, 12, 13]. Recently, in [14] the concept of PIO has been generalised to Multiple-Integral Observer (MIO), by replacing the integral action by a chain of integral actions. Thanks to these extra integral actions, the MIO is able to provide robust state estimations when the considered UI takes a polynomial form which is more general than the considered constant unknown input. Recently, other implementation schemes based on the MIO principle and their extension to Lipschitz nonlinear systems with single output are investigated in [15]. For generalised linear time-invariant state space systems (or descriptor system), the MIO design is addressed in [16, 17].

Moreover, nowadays the complexity of many physical systems is undoubtedly increased involving nonlinear dynamic behaviours which are governed by complex physical laws. Hence, the use of a single linear model for modelling the dynamic behaviour of such systems in the whole operating space is unsuited because a linear model provides only a local approximation in a small neighbourhood of an operating point (*local modelling*). On the other hand, rigorous modelling of complex systems in a wide operational range (*global modelling*) can be in practice very difficult and even if this modelling is possible the available model takes typically a nonlinear form which cannot be generally used in a systematic way for designing an observer.

However, a global representation of such systems can effectively be expressed by blending judiciously a set of *linear local models* which describes the dynamics of the system in some small region of the operating space. In this modelling framework, the obtained model is known as *local model network* or *multiple model* (see [18, 19, 20] and references therein). From a practical point of view, a multiple model is built by reducing the system complexity via a decomposition of its operating space in a finite number of operating zones within which the dynamics are characterised by a local model, also called *submodel*, with relatively simple structure, often linear. The global dynamic behaviour of the system is finally captured by considering the relative contribution of each submodel by means of a *weighting function* associated to each operating zone. In brief, multiple model approach provides a suitable framework for modelling nonlinear systems by an association of a set of submodels blended throughout an interpolation scheme.

It should be noted that the interpolation scheme used for blending the submodels plays an important role in this context. Indeed, the same set of submodels can be associated in various ways which provide several kinds of blended multiple models. However as pointed out in [21], two main structures of multiple models can be distinguished. In the first one, the multiple model is made up of *homogeneous submodels* in the sense that the submodels have the same structure and share the same state space across the operating space. In the second one, *heterogeneous submodels* can be used because their state spaces are decoupled. It should be noted that these two multiple model realisations don't have an identical dynamic behaviour.

The so-called *Takagi-Sugeno multiple model* is the main example of homogeneous multiple models. It has been initially proposed, in a fuzzy modelling framework, by Takagi and Sugeno [22] and in a operating regime-based modelling framework by Johansen and Foss [23]. This multiple model has been extensively considered in the literature for modelling, control and state estimation of nonlinear systems [18, 24, 25]. In this multiple model representation, the parameters of the submodels are combined by means of the weighting functions and then a common state vector appears in the dynamic equation of the overall model. It should be noted that this same multiple model structure is encountered in a number of quite diverse similar modelling frameworks such as piecewise linear model, polytopic models, hybrid systems, switched systems, markovian switching systems, etc. Despite different names, these approaches share the same modelling philosophy. In these different frameworks, the problem of estimating the state and the UI is tackled in several ways. For systems represented by Takagi-Sugeno multiple models, Luenberger observers including a sliding term to compensate the effect of the unknown inputs are used in [26]. For polytopic models, a polytopic unknown input observer design for providing actuator fault estimation in active fault tolerant control strategy is proposed in [27]. For markovian switching systems a finite memory observer is considered in order to cope with this estimation problem [28]. For hybrid systems an extension of the moving horizon estimation using a transformation of the original PieceWise Affine (PWA) system into Mixed Logical and Dynamical (MLD) systems is proposed in [29]. Nevertheless, these contributions are proposed on the basis of homogeneous local models across the operating space.

The second multiple model realisation , known as *decoupled multiple model*, can be used in order to take into consideration heterogeneous submodels. This model introduce some degree of flexibility in the modelling stage, since the dimension (e.g. the number of state variables) of each submodel can be adapted to the complexity of the system inside each operating zone and this constitutes the originality of the proposed approach (details are given in section 2). Nonlinear systems identification using decoupled multiple model is addressed in [30, 31, 32] and the control laws design in [33, 34, 35, 36]. It should be noted that the state estimation problem of a nonlinear system using this multiple model has been poorly considered in the literature [37, 38]. In [37] the state estimation problem is addressed in order to setting up a fault diagnosis strategy. However just a note on the estimation error convergence is proposed by the authors and no analytic proof of the convergence of the estimation error is given.

These works show the relevance and the successful implementation of this structure for state estimation.

The main contribution of this paper is an extension of MIO design for linear systems to nonlinear discrete-time systems represented by decoupled multiple models. The MIO is designed in order to provide both state and UI estimations by minimising the influence of external disturbances on the estimation error. An analytic proof of the convergence of the estimation errors is clearly established by using the well known Lyapunov theory. The robust $\mathcal{L}_2$ existence conditions of the MIO are expressed in the form of a set of linear matrix inequalities (LMIs) [39]. So far, the proportional and integral observer (PIO) design for linear discrete time systems seems only reported in some recent papers [40, 41]. Hence, the proposed results can also be used for designing a MIO in the single model linear case because single model and PIO are a particular case of multiple models and MIO.

The outline of this paper is as follows. The decoupled multiple model representation is presented in section 2. Preliminaries and the suggested MIO are presented in section 3. In section 4, robust $\mathcal{L}_2$ observer design is proposed and the gains of the observer are obtained by LMI optimization. The last section gives a simulation example to illustrate the effectiveness of the proposed approach.

Notations. *The following standard notations will be used.* $P > 0$ *(*$P < 0$*) denotes a positive (negative) definite matrix* P*;* X^T *denotes the transpose of matrix* X*,* I *is the identity matrix of appropriate dimension and* $diag\{A_1,...,A_n\}$ *stands for a block-diagonal matrix with the matrices* A_i *on the main diagonal. The* $\mathcal{L}_2-$*norm of a signal, quantifying its energy, is denoted and defined by* $\|e(k)\|_2^2 = \sum_0^\infty e^T(k)e(k)$*. Finally, we shall simply write* $\mu_i(\xi(k)) = \mu_i(k)$*.*

2 On the decoupled multiple model representation

Multiple model framework is an attractive way in the field of complex systems modelling because a large class of nonlinear dynamic behaviours can be captured using this representation. Note also that multiple model makes it possible the partial extension of some results obtained in the linear control theory to nonlinear systems avoiding specific analysis of the non-linearity of the system. In brief, multiple model offers good accuracy representation by means of an usable model.

The structure of the decoupled multiple model, firstly proposed in [21], is here slightly modified using a state space representation as follows:

$$x_i(k+1) = A_i x_i(k) + B_i u(k) + D_i \eta(k) + V_i w(k) \ , \tag{1a}$$

$$y_i(k) = C_i x_i(k) + E_i \eta(k) \ , \tag{1b}$$

$$y(k) = \sum_{i=1}^{L} \mu_i(\xi(k)) y_i(k) + W w(k) \ , \tag{1c}$$

where $x_i \in \mathbb{R}^{n_i}$ and $y_i \in \mathbb{R}^p$ are respectively the state vector and the output of the i^{th} submodel; $u \in \mathbb{R}^m$ is the known input vector, $\eta \in \mathbb{R}^l$ the UI vector, $y \in \mathbb{R}^p$ the measured output and $w \in \mathbb{R}^r$ a disturbance (noise, etc.). The matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$, $D_i \in \mathbb{R}^{n_i \times l}$, $V_i \in \mathbb{R}^{n_i \times r}$, $C_i \in \mathbb{R}^{p \times n_i}$, $E_i \in \mathbb{R}^{p \times l}$ and $W \in \mathbb{R}^{p \times r}$ are known and appropriately dimensioned.

Remark 1. *In this paper, the UI $\eta(k)$ is interpreted as an interesting variable to be estimated (e.g. faults, etc.) and $w(k)$ is a disturbance signal (e.g. noise, modelling errors, etc.).*

Remark 2. *The same UI appears in both local states and local outputs. However, different UIs in the local states and in the local outputs can be taken into consideration by choosing adequate structure of D_i, E_i and $\eta(k)$, for example, $D_i = [\bar{D}_i \quad 0]$, $E_i = [0 \quad E]$ and $\eta(k) = [\eta_a(k) \quad \eta_s(k)]^T$. Note that a common matrix E_i is used for modelling an UI acting on the multiple model output (e.g. a sensor fault).*

The so-called *decision variable* $\xi(k)$ is used in order to take into account the current operating point of the system. It is assumed known and real-time accessible, currently the inputs and/or measured variables of the system are employed as decision variable. The relative contribution of each submodel according to the operating point of the system is quantified by the weighting functions $\mu_i(\xi(k))$ which satisfy the following convex sum constraints:

$$\sum_{i=1}^{L} \mu_i(\xi(k)) = 1 \quad \text{and} \quad 0 \leq \mu_i(\xi(k)) \leq 1 \ , \quad \forall i = 1...L, \quad \forall k. \tag{2}$$

The role of the weighting functions is to allow a transition, often smooth, between the contribution of the submodels. Hence the contribution of several submodels can be taken into account at the same time because the weighting functions take intermediary values over the range 0 to 1. So the dynamic behaviour of the multiple model can therefore be considered as truly nonlinear instead of a piecewise linear behaviour.

As it can clearly be seen from equation (1), the submodels are run using a parallel scheme and the multiple model output is obtained via a weighted sum of the submodel outputs. Therefore, the submodels do not share the same state space and consequently their dimension (i.e. the number of state variables) and structure can be different across the operating space of the system. Hence, it can be expected that the decoupled multiple model accurately describes nonlinear systems with a relatively small number of parameters. Indeed, the use of heterogeneous submodels provides flexibility in the modelling stage because each submodel can be well adapted to the complexity of the system inside each operating zone. In a black box modelling, this feature can be used in order to cope with the so-called *curse of dimensionality* problem where the number of parameters needed for an accurate representation increases extremely rapidly as the order of the nonlinear dynamic system increases. This multiple model structure is then suited for a black box modelling of complex systems with variable structure in the operating range. Note that the local outputs $y_i(k)$ of the submodels are "artificial modelling signals" only used to provide a representation of the real system behaviour but can neither be related physically to the true system nor measured.

3 Preliminaries

Firstly, the aim of this section is to introduce a compact rewriting of the decoupled multiple model in order to reduce the further mathematical manipulation. Secondly, based on this new compact form, the proposed MIO is presented and the estimation errors are established. Finally, a brief discussion about sufficient conditions for the UI decoupled state estimation is proposed.

3.1 Compact representation of the multiple model

Notice that by using the following augmented state vector:

$$x(k) = \begin{bmatrix} x_1^T(k) & \cdots & x_i^T(k) & \cdots & x_L^T(k) \end{bmatrix}^T \in \mathbb{R}^n, n = \sum_{i=1}^{L} n_i \, , \tag{3}$$

the decoupled multiple model (1) can be rewritten in the following compact form:

$$x(k+1) = \tilde{A}x(k) + \tilde{B}u(k) + \tilde{D}\eta(k) + \tilde{V}w(k) \ \ , \tag{4a}$$

$$y(k) = \tilde{C}(k)x(k) + \tilde{E}(k)\eta(k) + Ww(k) \ \ , \tag{4b}$$

where

$$\tilde{A} = diag\{A_1 \cdots A_i \cdots A_L\} \ \ , \tag{5}$$

$$\tilde{B} = \begin{bmatrix} B_1{}^T \cdots B_i{}^T \cdots B_L{}^T \end{bmatrix}^T \ \ , \tag{6}$$

$$\tilde{D} = \begin{bmatrix} D_1{}^T \cdots D_i{}^T \cdots D_L{}^T \end{bmatrix}^T \ \ , \tag{7}$$

$$\tilde{V} = \begin{bmatrix} V_1{}^T \cdots V_i{}^T \cdots V_L{}^T \end{bmatrix}^T \ \ , \tag{8}$$

$$\tilde{C}(k) = \begin{bmatrix} \mu_1(k)C_1 \cdots \mu_i(k)C_i \cdots \mu_L(k)C_L \end{bmatrix} \ \ , \tag{9}$$

$$\tilde{E}(k) = \sum_{i=1}^{L} \mu_i(k)E_i \ \ . \tag{10}$$

The reader may have noticed that the time-varying matrix $\tilde{C}(k)$ can be rewritten as follows:

$$\tilde{C}(k) = \sum_{i=1}^{L} \mu_i(k)\tilde{C}_i \ \ , \tag{11}$$

where $\tilde{C}_i$ is a constant block matrix given by:

$$\tilde{C}_i = \begin{bmatrix} 0 \cdots C_i \cdots 0 \end{bmatrix} \tag{12}$$

such as the term C_i is found on the i^{th} block column of $\tilde{C}_i$.

3.2 Multi-Integral Observer presentation

Assuming that the UI acting on the system is modelled by a polynomial form of degree f in the variable k as follows

$$\eta(k) = Q_0 + Q_1 k + Q_2 k^2 + \cdots + Q_f k^f \quad . \tag{13}$$

From a practical point of view, a wide class of UIs can be modelled via a polynomial function (constant signal, ramps, etc.). Let us notice that the polynomial degree f of the unknown input is the only information supposed available, the coefficients Q_f of the polynomial are assumed unknown.

The following state observer on the basis of the model (4) and the UI (13) is proposed with the aim of providing a simultaneous estimation of the state and the UI:

$$\hat{x}(k+1) = \tilde{A}\hat{x}(k) + \tilde{B}u(k) + \tilde{D}\hat{\eta}_0(k) + K_p(y(k) - \hat{y}(k)) \quad , \tag{14a}$$

$$\hat{y}(k) = \tilde{C}(k)\hat{x}(k) + \tilde{E}(k)\hat{\eta}_0(k) \tag{14b}$$

coupled to the following UIO:

$$\hat{\eta}_i(k+1) = \hat{\eta}_i(k) + K_i(y(k) - \hat{y}(k)) + \hat{\eta}_{i+1}(k), \;\; i = 0, ..., q-1 \quad , \tag{15a}$$

$$\hat{\eta}_q(k+1) = \hat{\eta}_q(k) + K_q(y(k) - \hat{y}(k)) \quad , \tag{15b}$$

where $\hat{x}(k)$ and $\hat{\eta}_0(k)$ are the estimates of $x(k)$ and $\eta(k)$ respectively.

As can be shown in the figure 1, the UI estimation is obtained using the multi-integral strategy (in this figure the operator z^{-1} is the one step delay operator). Indeed, the estimation of the UI is performed using a recurrent schema given by a chain of integral actions. Hence, the inputs of the $i-1^{\text{th}}$ block are the output estimation error given by $y(k) - \hat{y}(k)$ and the output of the i^{th} block.

3.3 Definitions of the estimation errors

In this section, the state and UI estimation errors are studied to prove that $\hat{x}(k)$ and $\hat{\eta}_0(k)$ converge toward $x(k)$ and $\eta(k)$ respectively. In order to establish the estimation errors the *difference operator* must be introduced.

Definition 1 (Difference operator)**.** *The first difference of a function $\varphi(k)$ is a function defined by:*

$$\Delta\varphi(k) \triangleq \varphi(k+1) - \varphi(k) \quad . \tag{16}$$

The q^{th}-difference operator is given by:

$$\Delta^{(q)}\varphi = \Delta(\Delta^{(q-1)}\varphi(k)) \quad . \tag{17}$$

For example, the second difference of a function $\varphi(k)$ is given by:

$$\begin{aligned}\Delta^{(2)}\varphi &= \Delta(\varphi(k+1)-\varphi(k)) \ , \\ \Delta^{(2)}\varphi &= \varphi(k+2)-2\varphi(k-1)+\varphi(k) \ .\end{aligned}$$

Now, the state and UI estimation errors are defined by:

$$e(k) = x(k)-\hat{x}(k) \ , \tag{18}$$

$$\varepsilon_i(k) = \Delta^{(i)}\eta(k)-\hat{\eta}_i(k), \ \ i=0,...,q \ , \tag{19}$$

where $\Delta^{(0)}\eta(k)=\eta(k)$.

Remark 3. *The architecture of the proposed observer allows us to obtain an estimation of the state, the unknown input and its successive q^{th} differences (i.e. the coefficients of the polynomial) at the same time.*

Dynamic equations of these errors must be now established. Hence, dynamics of the state estimation error is given by:

$$e(k+1) = (\tilde{A}-K_p\tilde{C}(k))e(k)+(\tilde{D}-K_p\tilde{E}(k))\varepsilon_0(k)+(\tilde{V}-K_pW)w(k) \ . \tag{20}$$

The above equation is easily obtained with the help of equations (4), (14), (18) and (19) by considering $i=0$. Let us notice that the state estimation error is directly affected by the UI estimation error $\varepsilon_0(k)$ and the disturbance $w(k)$ acting on the system.

The dynamics of the UI estimation error, $\varepsilon_i(k)$ for $0 \leq i < q$, is obtained using (19) and (15):

$$\varepsilon_i(k+1) = \varepsilon_i(k)+\varepsilon_{i+1}(k)-K_i\tilde{C}(k)e(k)-K_i\tilde{E}(k)\varepsilon_0(k)-K_iWw(k), \ \ i=0,...,q-1 \ , \tag{21}$$

and finally the dynamics of $\varepsilon_q(k)$ is given by:

$$\varepsilon_q(k+1) = \varepsilon_q(k)-K_q\tilde{C}(k)e(k)-K_q\tilde{E}(k)\varepsilon_0(k)-K_qWw(k)+\Delta^{(q+1)}\eta(k) \ . \tag{22}$$

We can rewrite in a compact form the dynamics of the state estimation error, the UI estimation error and their successive differences using an augmented error vector given by:

$$\Sigma(k) = \begin{bmatrix} e^T(k) & \varepsilon_0^T(k) & \dots & \varepsilon_i^T(k) & \dots & \varepsilon_q^T(k) \end{bmatrix}^T \ , \tag{23}$$

as follows:

$$\Sigma(k+1) = (\Lambda-K\Gamma(k))\Sigma(k)+(V-KW)w(k)+\Phi\Delta^{(q+1)}\eta(k) \ , \tag{24}$$

where

$$\Lambda = \begin{bmatrix} \tilde{A} & \tilde{D} & 0 & 0 & 0 & \dots & 0 \\ 0 & \mathrm{I} & \mathrm{I} & 0 & 0 & \dots & 0 \\ 0 & 0 & \mathrm{I} & \mathrm{I} & 0 & \dots & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \mathrm{I} & \mathrm{I} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} \end{bmatrix}, K = \begin{bmatrix} K_p \\ K_0 \\ \vdots \\ K_i \\ \vdots \\ K_q \end{bmatrix}, \Phi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \mathrm{I} \end{bmatrix}, V = \begin{bmatrix} \tilde{V} \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \Gamma(k) = \begin{bmatrix} \tilde{C}(k)^T \\ \tilde{E}(k)^T \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}^T . \tag{25}$$

Notice that by using the definition (11) of $\tilde{C}(k)$ and the definition (10) of $\tilde{E}(k)$, the time-varying matrix $\Gamma(k)$ can be rewritten as:

$$\Gamma(k) = \sum_{i=1}^{L} \mu_i(k)\Omega_i . \tag{26}$$

where

$$\Omega_i = \begin{bmatrix} \tilde{C}_i & \tilde{E}_i & 0 & \dots & \dots & 0 \end{bmatrix} . \tag{27}$$

Finally, taking into consideration the previous matrix transformation, the equation (24) becomes:

$$\Sigma(k+1) = \Psi(k)\Sigma(k) + (V - KW)w(k) + \Phi\Delta^{(q+1)}\eta(k) , \tag{28}$$

$$\Psi(k) = \sum_{i=1}^{L} \mu_i(k)(\Lambda - K\Omega_i) . \tag{29}$$

Conceptually, the aim of the design is to determine an augmented gain matrix $K \in \mathbb{R}^{n+(q+1)l \times p}$ that guarantees the asymptotic convergence of the estimation error towards zero (detailed problem formulations are presented in the section 4).

3.4 Discussion about unknown input decoupling

It can be seen, from equation (28), that the estimation error is *totally decoupled* from the unknown input (i.e. the influence of the unknown input on the estimation error $\Sigma(k)$ is vanished) if and only if $\Delta^{(q+1)}\eta(k) = 0$. This conditions can be accomplished when the two following conditions are simultaneously satisfied:

1. the unknown input $\eta(k)$ takes a polynomial form of degree f in the variable k,

2. the number of integral actions q taken into account is at least equal to $f+1$.

Hence the number of integral actions q in the observer depends on the polynomial degree f taken into account for modelling the UI. Note that the minimal number of integral actions needed for ensuring the unknown input decoupling is fixed by the second condition. For example, if the UI is modelled by polynomial of degree zero

$\eta(k) = Q_0$ then only one integral action is necessary for ensuring $\Delta\eta(k) = 0$. If a polynomial of degree one is considered $\eta(k) = Q_0 + Q_1 k$ then two integral actions are necessary for ensuring $\Delta^2\eta(k) = 0$, and so on.

4 Problem formulations and main results

Three main problems can be examined in the purpose to simultaneously estimate the state and the unknown input using the proposed MIO:

1. **Problem 1.** Obtain conditions for ensuring both state and UI estimation error convergence, i.e. the observer stability, when $\Delta^{(q+1)}\eta(k) = 0$ and $w(k) = 0$.

2. **Problem 2.** Obtain conditions for both state and UI estimation error convergence by considering a perturbed multiple model i.e. $w(k) \neq 0$. This problem can be viewed as the robust observer design with respect to disturbances.

3. **Problem 3.** Obtain conditions for both state and UI estimation error convergence when the UI is not truly a polynomial form i.e. $\Delta^{(q+1)}\eta(k) \neq 0$ and by considering $w(k) \neq 0$.

Let us notice that in the multiple model framework, an unstable multiple model can be obtained by blending a set of stable submodels. Hence the stability assumption of the submodels does not guarantee the stability of the multiple model. Consequently, the individual stability of matrices $\Lambda - K\Omega_i$ is not sufficient in order to guarantee the stability of the time-varying $\psi(k)$ given by (29). Hence, a classic observer design for each submodel cannot be performed independently: the blending between the submodels must be taken into consideration in the observer design in order to ensure the convergence of the estimation error (23). Therefore, the analysis of these problems will be carried out in the time-domain with the help of the Lyapunov method and the proposed solutions for the observer design are derived in terms of LMIs [39].

4.1 Problem 1: convergence conditions of the estimation error

Convergence conditions of the estimation error, in the disturbance free case and with an UI perfectly decoupled, will be established under the following assumption:

Assumption 1. $w(k) = 0$ *and* $\Delta^{(q+1)}\eta(k) = 0$.

The following theorem presents the convergence conditions:

Theorem 1. *Consider the model* (4) *and assumption 1. There exists an observer* (14) *and* (15) *such that the state and the UI estimation error* (23) *asymptotically converges towards zero if there exist* $P = P^T > 0$ *and* M *such that:*

$$\begin{bmatrix} P & P\Lambda - M\Omega_i \\ (P\Lambda)^T - (M\Omega_i)^T & P \end{bmatrix} > 0, \quad i = 1 \ldots L \ . \tag{30}$$

The observer gains are given by $K = P^{-1}M$ *where* K *is defined by* (25).

Conceptually, the conditions of this theorem ensure the stability of the system defined in (28) for any blend between the submodel outputs and for any initial conditions.

Proof. Define the time dependent quadratic Lyapunov function by:

$$V(k) = \Sigma^T(k)P\Sigma(k), \tag{31}$$

where $P = P^T > 0$. The variation of the above function is given by:

$$\Delta V(k) \quad = \quad V(k+1) - V(k), \tag{32}$$

which must be negative in order to ensure the asymptotic convergence towards zero of the estimation error.

By considering the assumption 1 and (28) then (32) becomes:

$$\Delta V(k) \quad = \quad \Sigma^T(k)\{\Psi^T(k)P\Psi(k) - P\}\Sigma(k) \ \ , \tag{33}$$

that is a quadratic form in $\Sigma(k)$. The negativity of (33) is then ensured if:

$$\Psi^T(k)P\Psi(k) - P < 0 \ \ , \tag{34}$$

which can be rewritten, using the Schur complement and (29), as follows:

$$\sum_{i=1}^{L} \mu_i(k) \begin{bmatrix} P & P(\Lambda - K\Omega_i) \\ P(\Lambda - K\Omega_i) & P \end{bmatrix} > 0 \ \ . \tag{35}$$

Now, according to the convex sum properties (2), the above inequality is also verified if the following inequalities hold:

$$\begin{bmatrix} P & P(\Lambda - K\Omega_i) \\ (\Lambda - K\Omega_i)^T P & P \end{bmatrix} > 0 \quad , \quad i = 1 \ldots L \ \ . \tag{36}$$

Let us notice that the above inequalities do not take a LMI form in the variables P and K, therefore the classical LMI tools cannot be directly used. However, they become a LMI by setting $M = PK$ and the proof is completed. □

4.2 Problem 2: convergence conditions in the presence of disturbances

Now, model (4) is assumed to be affected by energy-bounded disturbances but the UI remains perfectly decoupled. In other words the following assumptions are made:

Assumption 2. *The disturbance* $w(k)$ *is such that* $\|w(k)\|_2^2 < \infty$.

Assumption 3. *The UI is such that* $\Delta^{(q+1)}\eta(k) = 0$.

The robust observer design problem can thus be formulated as finding a matrix gain K such that the objective signal $z(k)$ to be attenuated and defined by:

$$z(k) = H\Sigma(k) \ , \tag{37}$$

satisfies the following design objectives:

$$\lim_{k\to\infty} \Sigma(k) = 0 \quad \text{for} \quad w(k) = 0 \ , \tag{38a}$$

$$\|z(k)\|_2^2 \leq \gamma^2 \|w(k)\|_2^2 \quad \text{for} \quad w(k) \neq 0 \text{ and } z(0) = 0 \ , \tag{38b}$$

where γ is the $\mathcal{L}_2$ gain from $w(k)$ to $z(k)$. Notice that the convergence of the estimation error in the disturbance free case is ensured by (38a) and robust state estimation in presence of a disturbance is ensured by (38b). Finally, note that thanks to matrix H in (37), the attenuation level is guaranteed by considering partially or totally the components of the augmented error $\Sigma(k)$.

The following theorem presents robust convergence conditions:

Theorem 2 (Robust convergence conditions)**.** *Considerer the model* (4) *and assumptions 2 and 3. There exists an observer* (14) *and* (15) *ensuring the objective* (38) *if there exist* $P = P^T > 0$, M *and a scalar* $\gamma > 0$ *such that:*

$$\begin{bmatrix} -P & P\Lambda - M\Omega_i & PV - MW \\ (P\Lambda - M\Omega_i)^T & -P + H^T H & 0 \\ (PV - MW)^T & 0 & -\gamma^2 I \end{bmatrix} < \quad 0, \qquad i = 1 \ldots L \ , \tag{39}$$

for a prescribed matrix H. *The matrix gain is given by* $K = P^{-1}M$.

Remark 4. *The attenuation level* γ *can be minimized by considering* $\bar{\gamma} = \gamma^2$ *as an LMI variable to be minimized under the constraints* $\gamma > 0$, $P = P^T > 0$ *and* (39).

Proof. In order to ensure robust performances defined by (38), let us consider the quadratic Lyapunov function (31), its variation $\Delta V(k) = V(k+1) - V(k)$ and $\gamma > 0$ such that [39]:

$$\Delta V(k) < -z^T(k)z(k) + \gamma^2 w^T(k)w(k) \quad , \quad \forall k \ . \tag{40}$$

It is easily established that, by considering $V(0) = 0$, (40) implies $\|z(k)\|_2^2 < \gamma^2 \|w(k)\|_2^2$. Hence, robust performances (38) are achieved by satisfying (40). To that purpose, consider (28) and assumptions 2 and 3, the variation of the Lyapunov function is then given by:

$$\begin{aligned}\Delta V(k) &= \Sigma^T(k)\Psi^T(k)P\Psi(k)\Sigma(k) \\ &+ \Sigma^T(k)\Psi^T(k)P(V-KW)w(k) \\ &+ w^T(k)(V-KW)^T P\Psi(k)\Sigma(k) \\ &+ w^T(k)(V-KW)^T P(V-KW)w(k) - \Sigma^T(k)P\Sigma(k) \ ,\end{aligned} \tag{41}$$

which can be rewritten as:

$$\Delta V(k) = \Phi^T(k)\Omega(k)\Phi(k) \ , \tag{42}$$

where:

$$\Omega(k) = \begin{bmatrix} \Psi^T(k)P\Psi(k)-P & \Psi^T(k)P(V-KW) \\ (V-KW)^T P\Psi(k) & (V-KW)^T P(V-KW) \end{bmatrix} \ , \tag{43}$$

$$\Phi(k) = \begin{bmatrix} \Sigma^T(k) & w^T(k) \end{bmatrix}^T \ . \tag{44}$$

Finally, (40) is ensured if

$$\Phi^T(k)\left\{\Omega(k) + \begin{bmatrix} H^T H & 0 \\ 0 & -\gamma^2 \mathrm{I} \end{bmatrix}\right\}\Phi(k) < 0 \ , \tag{45}$$

which is a quadratic form in $\Phi(k)$. This inequality holds provided the expression inside the brackets is negative definite:

$$\begin{bmatrix} \Psi^T(k)P\Psi(k)-P+H^T H & \Psi^T(k)P(V-KW) \\ (V-KW)^T P\Psi(k) & (V-KW)^T P(V-KW)-\gamma^2 \mathrm{I} \end{bmatrix} < 0 \ , \tag{46}$$

which becomes using the Schur complement:

$$\begin{bmatrix} -P & P\Psi(k) & P(V-KW) \\ \Psi^T(k)P & -P+H^T H & 0 \\ (V-KW)^T P & 0 & -\gamma^2 \mathrm{I} \end{bmatrix} < 0 \ . \tag{47}$$

The proof is completed as in the previous case by using (29), the convex sum properties of $\mu_i(k)$ given by (2) and the change of variables $M = PK$. □

4.3 Problem 3: state and UI estimation when the $(q+1)^{\mathbf{th}}$-difference of the UI is not null

In this section, a perturbed decoupled multiple model without a total UI decoupling is considered (assumption 3 is not verified). Indeed, the $(q+1)^{\text{th}}$-difference of the unknown input is not assumed to be null but bounded by an unknown function $\Delta^{(q+1)}\eta(k) \leq \delta(k)$. Note that the solution of this problem can be useful in order to reduce the number of integral actions.

Assumption 4. *The UI satisfies $\Delta^{(q+1)}\eta(k) = \delta(k)$ with $\|\delta(k)\|_2^2 < \infty$.*

The estimation error equation (28) can be re-written as:

$$\Sigma(k+1) \quad = \quad \Psi(k)\Sigma(k) + \Pi\theta(k) \ , \tag{48}$$

where

$$\Pi = \begin{bmatrix} V - KW & \Phi \end{bmatrix} \text{ and } \theta(k) = \begin{bmatrix} w^T(k) & \delta^T(k) \end{bmatrix}^T \ . \tag{49}$$

Thus the unknown function $\delta(k)$ is considered as a disturbance. Now, this problem can be regarded in a similar way that the previous one. Hence the theorem 2 can be directly used by considering $\theta(k)$ as disturbance instead of $w(k)$.

The robust observer design problem can thus be formulated as finding a matrix gain K such that the objective signal $z(k)$ to be attenuated and defined by:

$$z(k) = H\Sigma(k) \ , \tag{50}$$

satisfies the following design objectives:

$$\lim_{k\to\infty} \Sigma(k) = 0 \quad \text{for} \quad w(k) = 0 \ , \tag{51a}$$

$$\|z(k)\|_2^2 \leq \gamma^2 \|\theta(k)\|_{2,Q}^2 \quad \text{for} \quad \theta(k) \neq 0 \text{ and } z(0) = 0 \ , \tag{51b}$$

where γ is the $\mathcal{L}_2$ gain from $\theta(k)$ to $z(k)$. Note that here a weighted L_2-norm defined by $\|\theta(k)\|_{2,Q}^2 = \sum_{0}^{\infty} \theta^T(k)Q\theta(k)$ where $Q > 0$ is employed in order to weight the relative importance given to w and δ .

The following theorem presents robust convergence conditions:

Theorem 3 (Robust convergence conditions)**.** *Considerer the model* (4) *and assumptions 2 and 4. There exists an observer* (14) *and* (15) *ensuring the objective* (51)*, if there exist $P = P^T > 0$, M and scalar $\gamma > 0$ such that:*

$$\begin{bmatrix} -P & P\Lambda - M\Omega_i & \tilde{\Pi} \\ (P\Lambda - M\Omega_i)^T & -P + H^T H & 0 \\ \tilde{\Pi}^T & 0 & -\gamma^2 Q \end{bmatrix} < \quad 0, \qquad i = 1 \ldots L \ ,$$

where

$$\tilde{\Pi} \quad = \quad \begin{bmatrix} PV - MW & \quad P\Phi \end{bmatrix} \ ,$$

for prescribed matrices H and $Q > 0$. The matrix gain is given by $K = P^{-1}M$.

Proof. The proof of this theorem is immediate from theorem 2. $\square$

5 Simulation example

In this section two examples are proposed in order to illustrate simultaneous state and UI estimation in presence of disturbances (problem 2) and in the case when the $(q+1)^{\text{th}}$-difference of the UI is not assumed to be null (problem 3).

Consider the model (4), made up of two different dimension submodels and involving two measured outputs and two unknown inputs. Here, the decision variable $\xi(k)$ is the input signal $u(k) \in [0,1]$ with sample time equal to $T_s = 0.01$. The weighting functions are obtained from normalised gaussian functions:

$$\mu_i(\xi(k)) \quad = \quad \omega_i(\xi(k))/\sum_{j=1}^{L} \omega_j(\xi(k)) \ , \tag{52}$$

$$\omega_i(\xi(k)) \quad = \quad \exp\left(-(\xi(k) - c_i)^2/\sigma^2\right), \ , \tag{53}$$

with $\sigma = 0.4$ and the centres $c_1 = 0.25$ and $c_2 = 0.75$.

The numerical values of the matrices of the submodels are as follows:

$$A_1 = \begin{bmatrix} -0.5 & -0.7 \\ 0.4 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.7 & 0.2 & 0.5 \\ 0.3 & -0.4 & -0.1 \\ -0.2 & -0.3 & 0.6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.4 \end{bmatrix} \ ,$$

$$C_1 = \begin{bmatrix} 0.7 & 0.4 \\ -0.5 & 0.3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.3 & 0.0 & 0.5 \\ -0.6 & -0.2 & -0.4 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.2 & 0.0 \\ 0.1 & 0.0 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0.1 & 0.0 \\ 0.2 & 0.0 \\ 0.3 & 0.0 \end{bmatrix} \ ,$$

$$E_1 = E_2 = \begin{bmatrix} 0.0 & 0.4 \\ 0.0 & 0.5 \end{bmatrix}, \qquad V_1 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & -0.2 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & -0.2 \\ -0.1 & 0.2 \end{bmatrix}, \quad W = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & -0.2 \end{bmatrix} \ .$$

The disturbance $w(k)$ and the UI $\eta(k)$ take respectively the following forms:

$$w(k) = \begin{bmatrix} 0.04\sin(40k) \\ 0.035\sin(60k) \end{bmatrix} \text{ and } \eta(k) = \begin{bmatrix} \eta_a(k) \\ \eta_s(k) \end{bmatrix} \ . \tag{54}$$

In this example the UI are faults acting on the system. Note the particular form of D_i, E_i and $\eta(k)$ which allows us to take into consideration UIs affecting independently the dynamics (e.g. an internal component failure) and the overall output of the multiple model (e.g. a sensor fault).

On the other hand, note that the eigenvalues of the submodels are given by :

$$\lambda_1 = \begin{bmatrix} -0.2 \pm 0.43i \end{bmatrix} \quad \text{and} \quad \lambda_2 = \begin{bmatrix} -0.83 & -0.16 & 0.50 \end{bmatrix} \ ,$$

thus, the dynamics of the submodels are different and the dynamic behaviour of the multiple model can be expected nonlinear. The input, outputs and the weighting functions of the multiple model in the UI-free case are plotted in figure 2. One should note that, at every time, both submodels are taken into consideration to compute the global model.

5.1 State and UI estimation in presence of disturbances: problem 2

In this section, faults $\eta_a(k)$ and $\eta_s(k)$ are defined by:

$$\eta_a(k) = \begin{cases} 0 & \text{if} \quad 0 \le kT_s \le 5, \\ 0.1333k - 0.6665 & \text{if} \quad 5 \le kT_s \le 6.5, \\ 0.2 & \text{otherwise,} \end{cases} \quad \text{and} \quad \eta_s(k) = \begin{cases} 0 & \text{if} \quad 0 \le kT_s \le 5, \\ -0.35 & \text{if} \quad 5 \le kT_s \le 8, \\ -0.15 & \text{otherwise.} \end{cases} \tag{55}$$

A total decoupling can be accomplished by using a MIO composed of two integral actions because η_a is piecewise polynomial of maximum degree equal to one and η_s is a piecewise constant signal.

The observer design is based on LMI conditions of theorem 2 by considering the attenuation level γ as unknown variable to be minimised (see remark 4). In this example, the objective signal $z(k)$ is the state estimation error of the submodels then $H = [\mathrm{I}_{(5\times5)} \quad 0_{(5\times4)}]$. Hence, only the $\mathcal{L}_2-$gain between the disturbances and the state estimation error will be attenuated in the observer design procedure.

The optimal solution satisfying conditions of theorem 2 is found by using the YALMIP interface [42] coupled to SeDuMi solver [43]. Thus, conditions of theorem 2 are fulfilled with:

$$K = \begin{bmatrix} -0.2863 & 0.4206 & 0.0760 & 0.3735 & 0.9123 & 2.1436 & 1.0075 & 0.0838 & 0.3212 \\ -0.4554 & -0.3321 & -0.6978 & -0.4836 & -1.4136 & -2.9376 & 1.2108 & -0.0309 & 0.2028 \end{bmatrix}^T , \tag{56}$$

with a minimal attenuation level given by $\gamma = 3.1623$.

State variables of the multiple model and state estimation errors provided by the proposed observer are plotted in figures 3 and 4. In the simulation, the initial conditions of the multiple model are not null and the initial conditions of the observers are null. The estimation error provided by the MIO is not globally affected by the UIs and remains globally bounded despite the disturbances and UIs acting on the system. Let us notice that the amplitudes of the obtained estimation errors are admissible with respect to the amplitudes of the state variables. Besides, the state estimation is punctually affected when the value of the UIs changes abruptly, for example at $kT_s = 5$ and $kT_s = 8$, obviously in this abrupt transition the UI is not a polynomial. However, the estimation errors provided by the proposed MIO have a good transient response and good robust performances.

The relationship between the $\mathcal{L}_2-$norms of $z(k)$ and $\gamma^2 w(k)$ is shown in figure 6. As clearly seen from this picture, the design objectives given by (38) are globally satisfied ($\|z(k)\|_2^2 < \gamma^2 \|w(k)\|_2^2$). Note that the proposed objectives are not well satisfied around the time origin due to the difference between the initial conditions of the multiple model and the MIO (clearly $z(0) \neq 0$).

On the other hand, it can be noted from figure 5 that the UIs are well estimated even if the influence of disturbances on the UI estimation errors is not taken into consideration in the observer design procedure. However, UI estimations can be improved by using a dedicated observer for estimating independently states and UI of the system or by adding a filtering stage. It can also be pointed out that the UI estimations are well performed despite that they appear simultaneously. From a diagnosis point of view, this feature can be very interesting. Indeed, the UI estimations given by the proposed observer can be used as fault symptoms of the system for fault detection, isolation and identification.

5.2 State and UI estimation when the $(q+1)^{\text{th}}$-difference of the UI is not null: problem 3

In this section, faults $\eta_a(k)$ and $\eta_s(k)$ are defined by:

$$\eta_a(k) = \begin{cases} 0 & \text{if} \quad 0 \leq kT_s \leq 2 \\ -0.1250k + 0.25 + 0.2\cos(4k) & \text{otherwise} \end{cases} \tag{57}$$

$$\eta_s(k) = \begin{cases} 0 & \text{if} \quad 0 \leq kT_s \leq 2.5 \\ 0.2 + 0.1\sin(3k) & \text{if} \quad 2.5 \leq kT_s \leq 7.5 \cdot \\ 0 & \text{otherwise} \end{cases} \tag{58}$$

The MIO comprises two integral actions. However, the UI decoupling cannot be accomplished as in the previous example because the UIs are not a strict polynomial of degree one. Here, only the $\mathcal{L}_2-$gain between the disturbances and the state estimation error will be attenuated in the observer design procedure then $H = [\mathrm{I}_{(5\times5)} \quad 0_{(5\times4)}]$ and the weight matrix $Q = \mathrm{I}_{(4\times4)}$.

Conditions of theorem 3 are fulfilled with:

$$K = \begin{bmatrix} 0.3078 & 0.5664 & 0.6752 & 0.6485 & 1.6586 & 4.6857 & 1.1085 & 0.7571 & 0.4754 \\ -0.2647 & -0.4227 & -0.5558 & -0.5017 & -1.3219 & -3.5949 & 1.4328 & -0.5357 & 0.3927 \end{bmatrix}^T, \tag{59}$$

with a minimal attenuation level given by $\gamma = 5.4772$.

On figure 7 the state estimation errors provided by the proposed observer are plotted. Let us notice that these estimation errors remain globally bounded and close to zero. The comparison of the UIs and their estimates are displayed on the figure 8. As clearly seen, the proposed observer provides good dynamic and robust performances even if a total decoupling of the UI cannot be accomplished.

6 Conclusion

In this paper, based on a decoupled multiple model representation of a nonlinear system, the design of a multi-integral unknown input observer has been proposed. The suggested multiple model is a promising alternative to the classic structures currently employed in the multiple model approach. Indeed, the dimension of each submodel can be different and some flexibility can be expected in black box modelling of complex system. The proposed observer is an extension of the well-known multi-integral observer that is able to provide both state and unknown input robust estimation. The effectiveness of the proposed approach is illustrated via a simulation example. The suggested observer can be used, as an extension of the classic generalized observer scheme, for the detection, isolation and identification of faults acting on the systems. Improvements to the proposed observer, in order to take into consideration a more general class of UI, provides promising prospects in the future.

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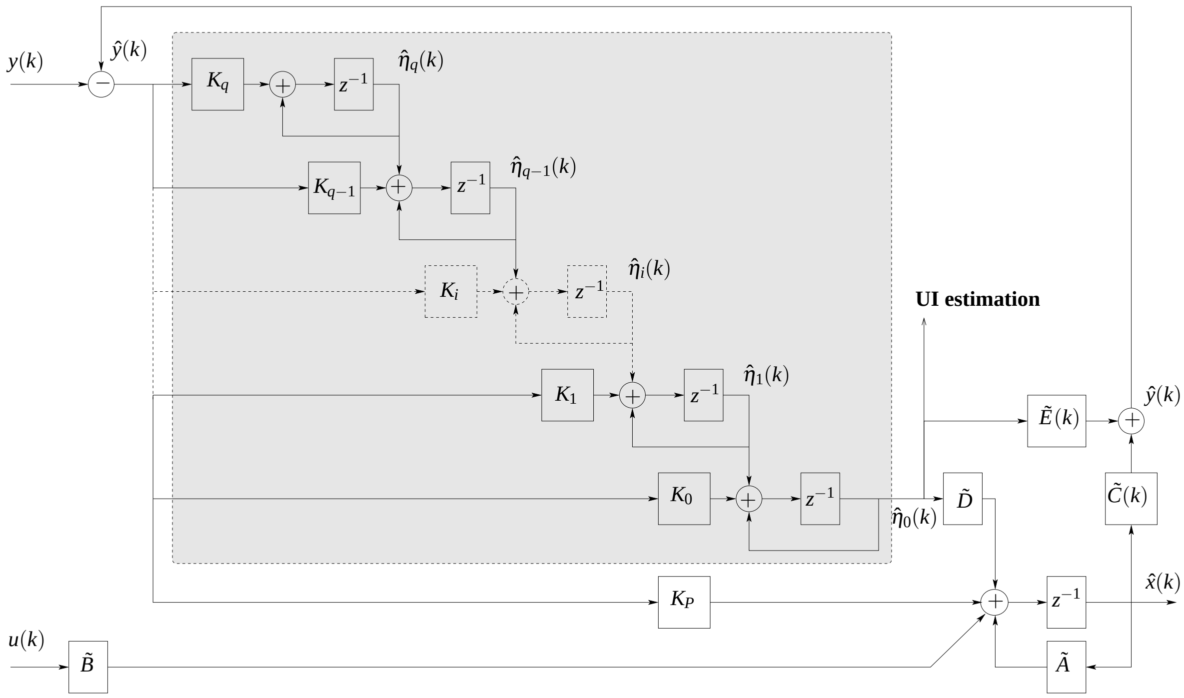

Figure 1: Architecture of the unknown input MPI observer

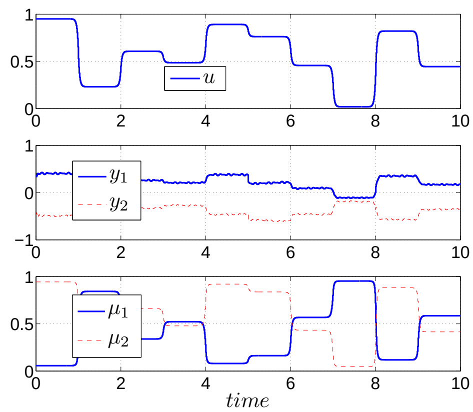

Figure 2: Input, outputs and weighting functions in the fault-free case

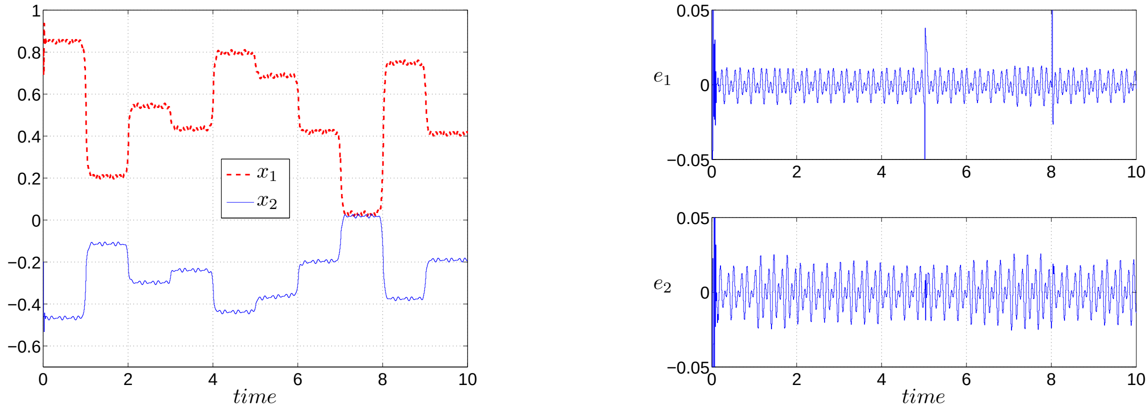

Figure 3: State variables and state estimation errors of submodel 1 (problem 2)

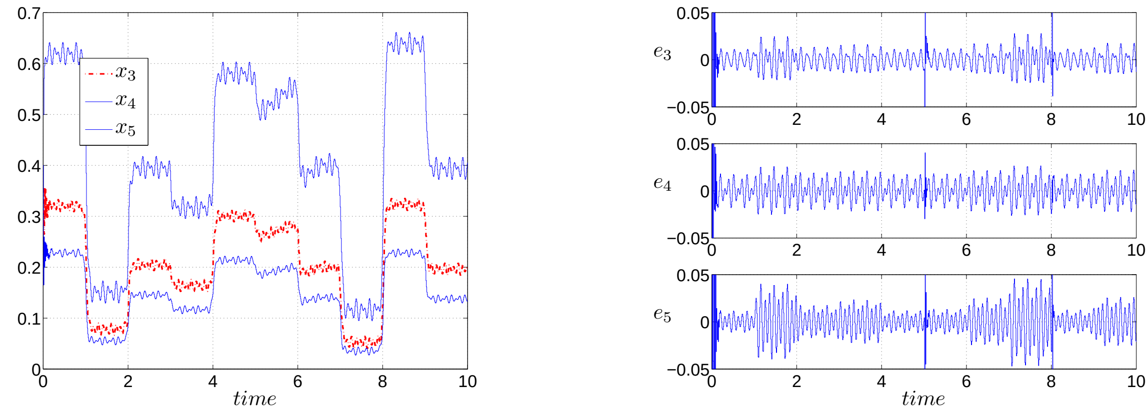

Figure 4: State variables and state estimation errors of submodel 2 (problem 2)

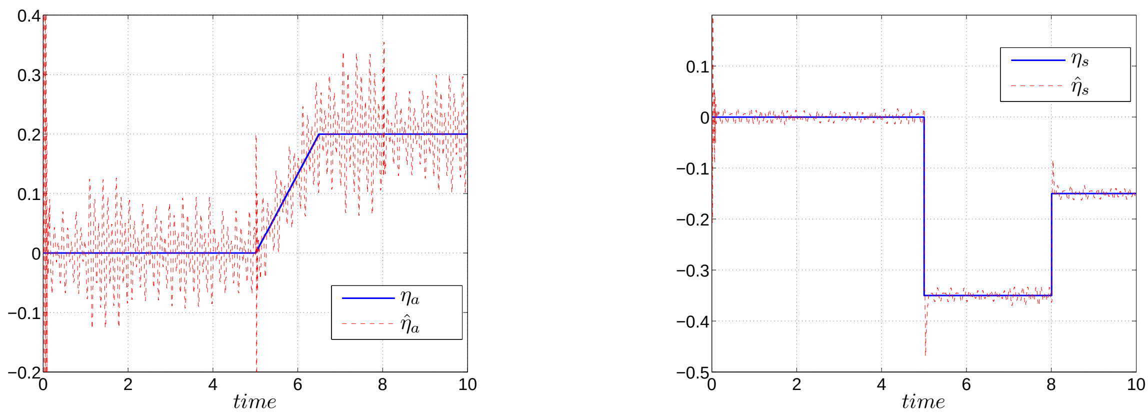

Figure 5: Unknown inputs η_a and η_s and their estimates (problem 2)

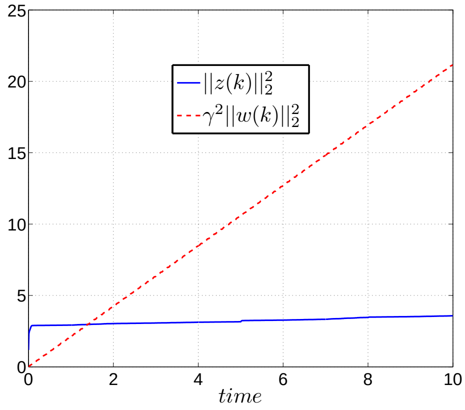

Figure 6: $\mathcal{L}_2-$norms of $z(k)$ and $\gamma^2 w(k)$ (problem 2)

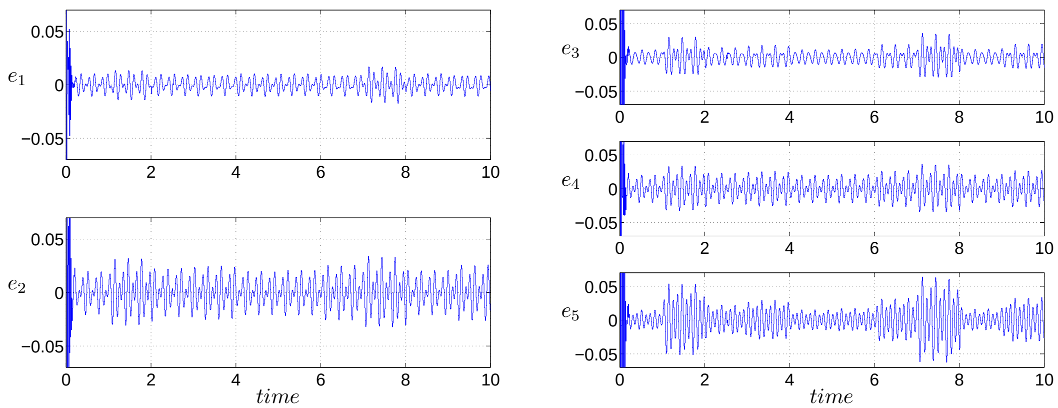

Figure 7: State estimation errors of submodels 1 and 2 (problem 3)

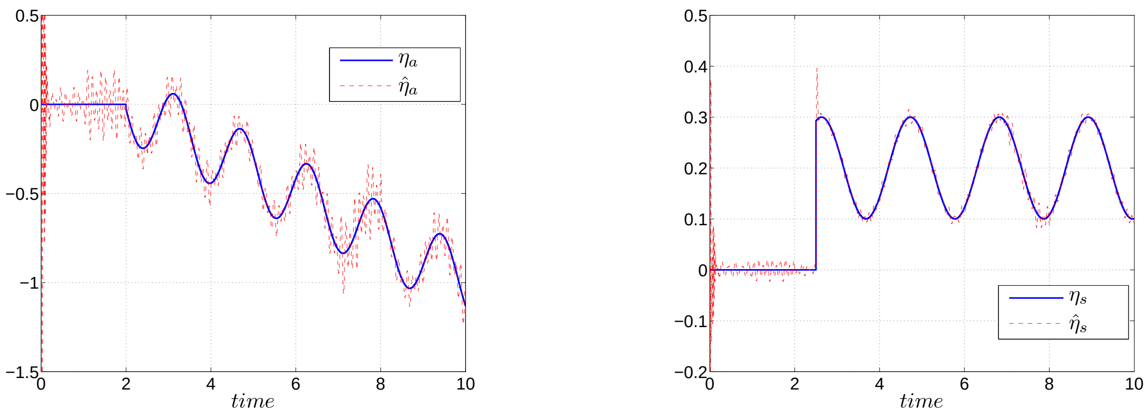

Figure 8: Unknown inputs η_a and η_s and their estimates (problem 3)