State estimation for non-linear systems using a decoupled multiple model

Rodolfo Orjuela,⋆ Benoît Marx, José Ragot and Didier Maquin

Centre de Recherche en Automatique de Nancy,
UMR 7039, Nancy-Université,
CNRS 2, Avenue de la Forêt de Haye,
Vandœuvre-lès-Nancy 54 516, France
E-mail: Rodolfo.Orjuela@ensem.inpl-nancy.fr
E-mail: Benoit.Marx@ensem.inpl-nancy.fr
E-mail: Jose.Ragot@ensem.inpl-nancy.fr
E-mail: Didier.Maquin@ensem.inpl-nancy.fr
⋆Corresponding author

Abstract: The multiple model approach is an elegant and a powerful tool for modelling real-world complex processes. In this modelling framework, a judicious combination of a set of submodels makes it possible to describe the behaviour of a non-linear system. Two different structures of multiple models can be distinguished according to whether the submodels share a common state vector *(Takagi-Sugeno multiple model)* or not (decoupled multiple model). This latter structure is an interesting alternative to the popular Takagi-Sugeno multiple model because different orders of submodels can be considered. The decoupled multiple model is nowadays increasingly used to perform the identification and the control of non-linear systems. However, to our knowledge, the state estimation problem of non-linear systems represented by this structure is not thoroughly investigated. The present paper deals with this worthwhile problem.

Keywords: state estimation; non-linear systems; multiple model approach; fault detection.

Biographical notes: Rodolfo Orjuela received his Masters in Artificial Intelligence, Pattern recognition and Robotics from the Paul Sabatier University (Toulouse) in 2004 and another Masters in Automatic control from the University of Bordeaux in 2005. He is a PhD student at the 'Institut National Polytechnique de Lorraine' (INPL), Nancy, France. His research activities, within Automatic Control Research Center of Nancy (CRAN), include the surveillance based on multiple model approach of non-linear systems.

Benoît Marx received his Engineer's Degree from the Ecole Centrale de Nantes (France) and another Masters from the University of Nantes, both in System Control, in 2000, and his PhD in Automatic Control from the 'Institut National Polytechnique de Grenoble' in 2003. He is an Assistant Professor at the 'Institut National Polytechnique de Lorraine' (INPL) since 2004 and member of the Automatic Control Research Center of Nancy (CRAN). His topic of interest is mainly the estimation and the diagnosis of descriptor and/or multiple model system.

José Ragot received his Engineer's Degree with specialisation in Control from the Ecole Centrale de Nantes (France) in 1969. Then, he joined the University of Nancy (France) where he received Masters in 1970. In 1973 he obtained his PhD and in 1980 the 'Diplôme de Doctorat-es-Science'. Since 1985, José Ragot is a full Professor at the 'Institut National Polytechnique de Lorraine' (INPL, National Polytechnic Institute of Lorraine) and a researcher in the Automatic Control Research Center of Nancy (CRAN). His fields of interest includes data validation, process diagnosis, fault detection and isolation, modelling and identification.

Didier Maquin received his PhD in Electrical Engineering in November 1987 and a so-called 'Habilitation à Diriger des Recherches' in 1997. Since 2003, he is a full Professor at the 'Institut National Polytechnique de Lorraine' (INPL, National Polytechnic Institute of Lorraine). Member of the Automatic Control Research Center of Nancy (CRAN), he is the scientist responsible of a research group dealing with dependability and diagnosis of industrial systems. He also coordinates a French national research group which is interested in the problems of maintenance, reliability, dependability and monitoring of industrial systems. His research interests are in modelling, supervision, fault detection and isolation of complex systems.

1   Introduction

A mathematical model is frequently necessary in order to cope with the state estimation problem in many engineering and science processes. An analytical model, often non-linear, can be obtained when the nature of the process is well understood. However, such non-linear models may be unusable in practice because the direct extension of conventional linear control tools, for designing a control law or setting up a diagnosis strategy, is not always possible.

Assuming that the considered process evolves around an operating point, a linear model can be identified via a conventional linear identification. Hence, the obtained linear model can be handled with the classical linear system theory. However, assuming that the process only evolves in the neighbourhood of a single operating point can be in practice, a serious limitation. Consequently, a single linear model is not able to describe the whole behaviour of the process in the operating range.

According to the foregoing assumption building a model both accurate and usable in practice, for a given process, needs the completion of two opposite objectives. New modelling techniques have been developed in order to cope with these difficulties. The multiple model approach, often related to the operating regime decomposition (Murray-Smith and Johansen, 1997), is among these techniques.

The backbone of the multiple model approach is the decomposition of the operating space of a non-linear system into a finite number of operating zones. Each operating zone is characterised by a submodel, often a linear model. The validity of each submodel is defined via a *weighting function* that ranges between zero and one. According to the zone where the non-linear system evolves, the output of each submodel is more or less requested in order to describe the whole behaviour of the non-linear system. Finally, the approximation of the whole behaviour of the non-linear system is achieved by a weighted combination (a blend) of the submodel contributions. (see Leith and Leithead, 2000) and the references therein for an extensive overview about different approaches based on this same principle.

It is important to notice that a large class of non-linear systems can efficiently be approximated by a multiple model, with a wished accuracy, by increasing the number of submodels and by a parametric optimisation of the weighting functions. Therefore, the multiple model can be considered as an *universal approximation tool* of non-linear systems. In other respects, most of the analysis tools available for linear systems can be partially extended to the analysis of non-linear systems represented by a multiple model if the submodels are linear and for particular weighting functions. These characteristics set the multiple model approach as an elegant tool with several uses in the fields of identification, control and diagnosis of a large class of complex systems.

As mentioned above, the output of the multiple model is carried out by taking judiciously into account the contribution of each submodel. Two different structures can be distinguished in order to provide the combination between the submodels (Filev, 1991). In the first structure, the submodels share the same state vector (*Takagi-Sugeno multiple model*); in the second one, the submodels are decoupled and their state vectors are different (*decoupled multiple model*).

The Takagi-Sugeno multiple model has been largely popularised in this modelling framework. The state estimation problem has been widely addressed (Chadli et al., 2005; Guerra et al., 2006; Rhee and Won, 2006; Tanaka and Sugeno, 1990; Ting, 2006) in the perspective of designing a stabilising control law. The classically used state estimator is an extension of the proportional (Luenberger) observer. However, some other classes of state estimators have been proposed, for instance, sliding mode observers (Bergstern et al., 2002) and unknown input observers (Akhenak et al., 2004). The state estimation can also be useful for setting up a diagnosis strategy based on the *residual signals* evaluation (Lopez-Toribio et al., 2000; Rodrigues et al., 2006).

Despite the fact that the decoupled multiple model has been less investigated than the Takagi-Sugeno multiple model, this kind of multiple model seems to be an interesting alternative. Indeed, the major interest of the decoupled multiple model lies in the fact that the order of each submodel can be different. Therefore, the identification capacity of this multiple model is increased and it can be used to perform the identification (Orjuela et al., 2006; Thiaw et al., 2007; Venkat et al., 2003) and the control (Gatzke and Doyle III, 1999; Gawthrop, 1995; Gregorcic and Lightbody, 2000) of highly non-linear systems. More recently, (Kanev and Verhaegen, 2006) have proposed a method for evaluating the submodel weights of this kind of multiple model. The previously quoted works have illustrated a successful implementation of this structure and have shown its relevance. However, to the best of the authors' knowledge, the state estimation problem of a non-linear system using a decoupled multiple model has been poorly considered in the literature.

This paper proposes a method for designing an observer for a non-linear system modelled by a decoupled multiple model. The outline of this paper is as follows. Section 2 introduces the two multiple model structures, namely the *Takagi-Sugeno multiple model* and the *decoupled multiple model*. Stability of decoupled multiple model is investigated in Section 3. In Section 4, the state estimation problem is considered. Sufficient conditions are given, in Linear Matrix Inequalities (LMIs) terms, in order to ensure the convergence of the estimation error. Eigenvalue assignment problem, in a specific subregion of the complex plane, for the proposed observer is also discussed in this section. Finally, in Section 5, the state estimation of a decoupled multiple model and its application to sensor fault detection and isolation is illustrated through a simple simulation example.

Notation: The following notations will be used all along this paper. $P > 0$ ($P < 0$) denotes a positive (negative) definite matrix P; X^T denotes the transpose of matrix X and I is the identity matrix of appropriate dimension. We shall simply write $\mu_i(\xi(t)) = \mu_i(t)$.

2   Multiple model structures

The multiple model structures can be divided into two main categories: Takagi-Sugeno multiple model and decoupled

multiple model. In the first case, the submodels use a common state vector; in the second one, the submodels have a different state vector.

Once the multiple model structure has been chosen, among the above mentioned structures, a non-linear system can be identified using available identification tools for estimating multiple model parameters (see Babuska and Verbruggen, 2003; Gasso et al., 2001; Murray-Smith and Johansen, 1997; Orjuela et al., 2006; Venkat et al., 2003 for further details about these techniques).

2.1 Takagi-Sugeno multiple model

The Takagi-Sugeno multiple model, also called multiple model with a common state, is widely used in the multiple model analysis and synthesis (Murray-Smith and Johansen, 1997). The structure of this multiple model is defined by (see Figure 1):

$$\dot{x}(t) = \sum_{i=1}^{L} \mu_i(\xi(t))\big\{A_i x(t) + B_i u(t)\big\} \qquad (1)$$

$$y(t) = \sum_{i=1}^{L} \mu_i(\xi(t))C_i x(t)$$

where $x \in \mathbb{R}^n$ is the common state vector of the submodels, $u \in \mathbb{R}^m$ the input, $y \in \mathbb{R}^p$ the output, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$ and $C_i \in \mathbb{R}^{p \times n}$ are constant matrices.

Figure 1 Takagi-Sugeno multiple model architecture
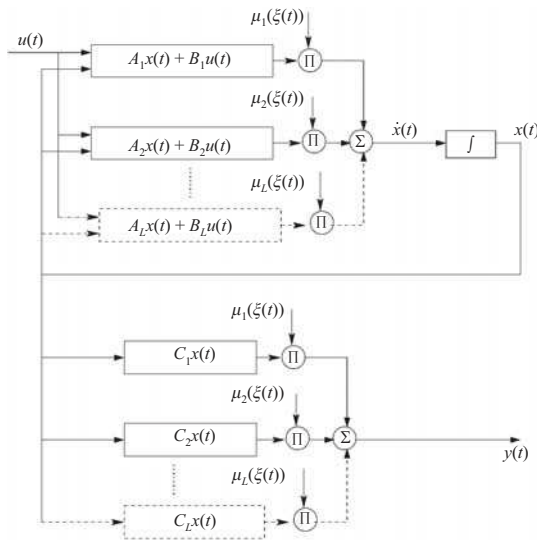
Moreover, the $\mu_i(\xi(t))$ are the *weighting functions* that ensure smooth transitions between the submodels. They have the following properties:

$$\sum_{i=1}^{L} \mu_i(\xi(t)) = 1, \quad \forall t \qquad (2a)$$

$$0 \le \mu_i(\xi(t)) \le 1, \quad \forall i = 1...L, \forall t \qquad (2b)$$

where $\xi(t)$ is the decision variable that depends on the measurable signals, for example, the input and/or the output of the system.

In this multiple model framework, the relative contribution of each submodel is taken into account thanks to a weighted

sum of submodel parameters. Indeed, this multiple model can be regarded as a system whose parameters vary with time as follows:

$$\dot{x}(t) = \tilde{A}(t)x(t) + \tilde{B}(t)u(t) \qquad (3)$$

$$y(t) = \tilde{C}(t)x(t)$$

where

$$\tilde{A}(t) = \sum_{i=1}^{L} \mu_i(\xi(t))A_i, \qquad \tilde{B}(t) = \sum_{i=1}^{L} \mu_i(\xi(t))B_i,$$

$$\tilde{C}(t) = \sum_{i=1}^{L} \mu_i(\xi(t))C_i$$

The above structure is frequently found in several modelling approaches: piecewise linear model (Sontag, 1981), radial basis functions networks (Fritzke, 1997), fuzzy models (Tanaka and Sugeno, 1990), Linear Parameter Varying (LPV) models (Shamma and Athans, 1991), etc. Differences between these approaches are due to the choice and the interpretation of the weighting functions on the one hand and to the structure of the submodel employed on the other hand. However, despite different names, these approaches have a similar mathematical basis.

2.2 Decoupled multiple model

Filev (1991) proposes another multiple model structure based on a parallel interconnection of the submodels (see Figure 2). Here, this structure is slightly modified using a state representation as follows:

$$\dot{x}_i(t) = A_i x_i(t) + B_i u(t)$$

$$y_i(t) = C_i x_i(t) \qquad (4)$$

$$y(t) = \sum_{i=1}^{L} \mu_i(\xi(t))y_i(t)$$

where $x_i \in \mathbb{R}^{n_i}$ and $y_i \in \mathbb{R}^p$ are the state vector and the output vector of the ith submodel respectively and where u, y, ξ, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$ and $C_i \in \mathbb{R}^{p \times n_i}$ have been defined in the previous section.

Here, the contribution of each submodel is taken into account via a weighted sum of the submodel outputs (blending in the static equation). Indeed, as seen from Figure 2, this multiple model is built by a parallel interconnection of Wiener model (i.e. a linear model followed by a non-linear function). Therefore, each submodel evolves independently in its own state space depending on the input control and its initial state.

Thanks to this fact, the dynamics of the submodels are completely decoupled and consequently the dimension of the state vector x_i of each submodel can be different (of course the output vectors dimensions must be identical). Therefore, this structure is well adapted for modelling strongly non-linear systems whose structure varies with the operating regime, for example when the complexity of the dynamic behaviour is not uniform in the operating range.

It should be noted that the fundamental difference between the decoupled multiple model schema and the popular schema employed in a *gain scheduling strategy* lies in the

fact that in the latter case only one model is used at the same time (switching). Consequently, a crisp transition between submodels appears, unlike the proposed schema where a smooth transition between the submodels may be ensured via the weighting functions.

Figure 2 Decoupled multiple model architecture
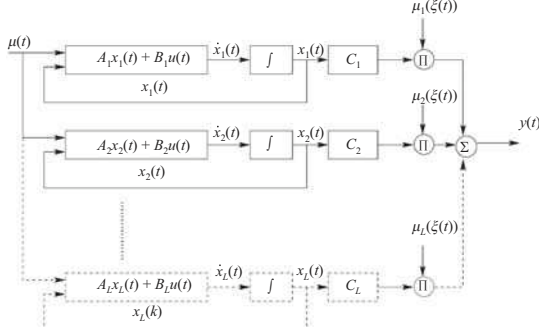
3 Stability of multiple models

In this section, a basic sufficient condition for ensuring the stability of the Takagi-Sugeno multiple model is reminded and the stability of the decoupled multiple model is investigated.

3.1 Takagi-Sugeno multiple model

The stability of the Takagi-Sugeno multiple model (3) is established with the help of the Lyapunov direct method (Tanaka and Sugeno, 1990). The asymptotic stability is guaranteed if there exists a matrix P symmetric, positive definite such that:

$$A_i^T P + P A_i < 0, \quad i = 1, 2, ..., L \tag{5}$$

The above condition is only a *sufficient condition*. It is interesting to note that the stability of the Takagi-Sugeno multiple model is not guaranteed by the individual stability of each submodel. Indeed, the stability of this multiple model depends on the existence of a common positive definite matrix P for all submodels.

3.2 Decoupled multiple model

By using an augmented state vector, Equation (4) may be rewritten in the following compact form:

$$\dot{x}(t) = \tilde{A} x(t) + \tilde{B} u(t)$$
$$y(t) = \tilde{C}(t) x(t) \tag{6}$$

where

$$\tilde{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & A_i & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & A_L \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 \\ \vdots \\ B_i \\ \vdots \\ B_L \end{bmatrix}$$

$$\tilde{C}(t) = [\mu_1(t) C_1 \cdots \mu_i(t) C_i \cdots \mu_L(t) C_L]$$

$$x(t) = [x_1^T(t) \cdots x_i^T(t) \cdots x_L^T(t)]^T \in \mathbb{R}^n, \ n = \sum_{i=1}^{L} n_i$$

Theorem 1: *The decoupled multiple model (4) is stable if and only if all of the eigenvalues of the matrix $\tilde{A}$ lie in the open left half-plane, that is, if and only if all of the submodels are stable; in other words, if there exists $P_i = P_i^T > 0$ such that:*

$$A_i^T P_i + P_i A_i < 0, \quad i = 1, 2, ..., L \tag{7}$$

Proof: The stability of the decoupled multiple model can be easily established by investigating the eigenvalues of the constant matrix $\tilde{A}$. The matrix $\tilde{A}$ is a block diagonal matrix; therefore the eigenvalues of this matrix are in the open left-half complex plane if and only if all eigenvalues of all matrices A_i are also in the left-half complex plane.

Let us notice that the Theorem 2 provides a *necessary and sufficient condition*.

To sum up, the stability of the decoupled multiple model is equivalent to the stability of all the submodels, in contrast to the stability of the Takagi-Sugeno multiple model that depends on the solution of a set of LMIs (Boyd et al., 1994).

4 State estimation

In this section, the state estimation problem of a non-linear system modelled by a decoupled multiple model is considered.

It should be reminded that, in the proposed schema, the output of the system is approximated by a weighted sum of the submodel outputs. Therefore, the observer design for each submodel cannot be accomplished with conventional tools. Indeed, in our approach, it is necessary to take into account the blend between the submodel outputs in order to guarantee the convergence of the estimation error for any blend.

Consequently, the Lyapunov second method is employed in order to establish sufficient conditions, in terms of a set of LMIs, for ensuring the estimation error convergence.

4.1 Observer structure

The state estimation is achieved with the help of the following proportional gain observer:

$$\dot{\hat{x}}_i(t) = A_i \hat{x}_i(t) + B_i u(t) + K_i(y(t) - \hat{y}(t))$$
$$\hat{y}_i(t) = C_i \hat{x}_i(t) \tag{8}$$
$$\hat{y}(t) = \sum_{i=1}^{L} \mu_i(\xi(t)) \hat{y}_i(t)$$

where $\hat{x}_i \in \mathbb{R}^{n_i}$ is the state estimation vector of the ith submodel, $y \in \mathbb{R}^p$ the output of the multiple model, $\hat{y} \in \mathbb{R}^p$ the estimated output and $K_i \in \mathbb{R}^{n_i \times p}$ the gain of the ith observer to be determined. It is assumed in the sequel that the decision variable $\xi(t)$ is measurable.

The observer Equations (8) can also be written using the partitioned matrices already defined by Equation (6):

$$\dot{\hat{x}}(t) = \tilde{A} \hat{x}(t) + \tilde{B} u(t) + \tilde{K}(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = \tilde{C}(t) \hat{x}(t) \tag{9}$$

where the blocks of the partitioned matrix $\tilde{K}$ are the gain matrices K_i of each observer:

$$\tilde{K} = [K_1^T \cdots K_i^T \cdots K_L^T]^T \in \mathbb{R}^{n \times p} \tag{10}$$

The design of the observer consists in finding a matrix $\tilde{K}$ ensuring that the estimation error given by:

$$e(t) = x(t) - \hat{x}(t) \tag{11}$$

tends to zero for any combination between the submodel outputs and for any initial conditions.

From Equation (11) and by using Equations (6) and (9), the time derivative of the estimation error is given by:

$$\dot{e}(t) = A_{\text{obs}}(t)e(t) \tag{12}$$

where

$$A_{\text{obs}}(t) = \tilde{A} - \tilde{K}\tilde{C}(t) \tag{13}$$

The solution of the differential Equation (12) gives the evolution of the estimation error. However, the analytical solution of this differential equation is not easily established due to the time-varying matrix $A_{\text{obs}}(t)$. The Lyapunov second method is employed in order to cope with this difficulty. This method allows us to analyse the estimation error of the observer without the explicit integration of the differential Equation (12).

4.2 Convergence of the estimation error

The exponential convergence of the estimation error (α-stability of the observer) is a way to ensure a convergence velocity of the estimation error via a *decay rate* and to improve dynamic performances of the observer. The following theorem provides sufficient conditions for ensuring the exponential convergence of the estimation error.

Theorem 2: *Consider the decoupled multiple model (6) and the observer* (9). *The exponential convergence of the estimation error is guaranteed if there exists a symmetric and positive definite matrix P, a matrix G and a positive scalar α such that:*

$$(\tilde{A} + \alpha I)^T P + P(\tilde{A} + \alpha I) - (G\tilde{C}_i)^T - G\tilde{C}_i < 0$$

for $i = 1...L$, where $\tilde{C}_i = [\ 0 \cdots C_i \cdots 0\ ]$.
The observer gain is given by $\tilde{K} = P^{-1}G$.

Proof: The proof of this theorem is performed by using the following quadratic Lyapunov function:

$$V(t) = e^T(t)Pe(t), \quad P > 0 \quad P = P^T \tag{14}$$

The exponential convergence of the estimation error is guaranteed if:

$$\exists P = P^T > 0, \ \alpha > 0 : \dot{V}(t) + 2\alpha V(t) < 0 \tag{15}$$

where α is called the *decay rate*. Indeed, the solution of the Equation (15) is given by:

$$\dot{V}(t) \leq V(0)\exp(-2\alpha t), \ \forall t \geq 0 \tag{16}$$

Due to $\lambda_{\min}(P)\|e(t)\|^2 \leq V(t) \leq \lambda_{\max}(P)\|e(t)\|^2$, the norm of the estimation error can be bounded by:

$$\|e(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\exp(-\alpha t)\|e(0)\|, \ \forall t \geq 0 \tag{17}$$

The derivative of (14) with respect to time yields:

$$\dot{V}(t) = \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) \tag{18}$$

that becomes by employing (Gatzke and Doyle, 1999):

$$\dot{V}(t) = e^T(t)\left\{A_{\text{obs}}^T(t)P + PA_{\text{obs}}(t)\right\}e(t) \tag{19}$$

Using (14) and (19), the inequality (15) becomes:

$$e^T(t)\left\{A_{\text{obs}}^T(t)P + PA_{\text{obs}}(t) + 2\alpha P\right\}e(t) < 0 \tag{20}$$

that is a quadratic form in $e(t)$. Therefore, the above inequality is satisfied if the following inequality holds:

$$A_{\text{obs}}^T(t)P + PA_{\text{obs}}(t) + 2\alpha P < 0 \tag{21}$$

that is a *sufficient condition* for ensuring the exponential convergence of the estimation error.

Thanks to the weighting functions property 2(a), it should be noted that the matrix $A_{\text{obs}}(t)$ can be rewritten as follows:

$$A_{\text{obs}}(t) = \sum_{i=1}^{L} \mu_i(t)\phi_i \tag{22}$$

$$\phi_i = \tilde{A} - \tilde{K}\tilde{C}_i \tag{23}$$

where the constant bloc matrix $\tilde{C}_i$ is given by:

$$\tilde{C}_i = \begin{bmatrix} 0 & \cdots & C_i & \cdots & 0 \end{bmatrix} \tag{24}$$

Hence, using the previous definition of matrix $A_{obs}(t)$, the inequality (21) becomes:

$$\sum_{i=1}^{L} \mu_i(t)\phi_i^T P + P \sum_{i=1}^{L} \mu_i(t)\phi_i + 2\alpha P < 0 \tag{25}$$

Considering the weighting functions property 2(a), the inequality (25) is also equivalent to:

$$\sum_{i=1}^{L} \mu_i(t)\phi_i^T P + P \sum_{i=1}^{L} \mu_i(t)\phi_i + \sum_{i=1}^{L} \mu_i(t)2\alpha P < 0 \tag{26}$$

The inequality (26) holds if the following inequalities are satisfied:

$$\phi_i^T P + P\phi_i + 2\alpha P < 0, \quad i = 1, ..., L \tag{27}$$

or equivalently by substituting ϕ_i by its definition (23):

$$\tilde{A}^T P + P\tilde{A} - (\tilde{K}\tilde{C}_i)^T P - P\tilde{K}\tilde{C}_i$$
$$+2\alpha P < 0, \ i = 1, ..., L \tag{28}$$

Let us notice that the above inequalities are bilinear in $\tilde{K}$ and P. Therefore, it is not possible to solve them directly using standard convex optimisation algorithms. However, the following change of variable:

$$G = P\tilde{K}$$

can be useful in order to linearise (28). Finally, the matrix inequalities (28) become:

$$\tilde{A}^T P + P\tilde{A} - (G\tilde{C}_i)^T - G\tilde{C}_i + 2\alpha P < 0, \quad i = 1, ..., L \tag{29}$$

that are linear inequalities in P and G. Now, a solution can be found using classical LMI tools. The proof of Theorem 2 is completed by factoring the terms in P.

Remark 1: *Exponential convergence error is a strong form of convergence; it implies asymptotic convergence. Indeed, the asymptotic convergence of the estimation error is obtained by considering a decay rate equal to zero ($\alpha = 0$) in Theorem 2.*

4.3   Eigenvalues assignment

Dynamic performances of the estimation error (e.g. convergence velocity, damping, etc.) can be enforced by assigning the eigenvalues of the matrix $A_{\text{obs}}(t)$ in a specific region of the complex plane.

Let us notice that the exponential convergence of the estimation error is carried out by assigning the eigenvalues of the matrix $A_{\text{obs}}(t)$ in the shifted left-half plane. Hence, the convergence velocity of the estimation error is enforced. However, in this approach, only the real parts of the eigenvalues of the matrix $A_{\text{obs}}(t)$ are assigned (the real parts of eigenvalues of $A_{\text{obs}}(t)$ are lower than $-\alpha$). Consequently, dynamics with strong oscillations can appear because the imaginary part of the eigenvalues is unbounded.

In order to avoid this phenomenon, the eigenvalues of the matrix $A_{\text{obs}}(t)$ can be assigned in a particular region S of the complex plane. Chilali and Gahinet (1996) have proposed a general characterisation for eigenvalues clustering in subregions of the complex plan in terms of LMIs.

The region S defined by the intersection of a disk (centred at the origin, with radius r) and the shifted left-half plane (i.e. $S(\alpha, r) = \{z \in \mathbb{C}, |z| < r, \Re e(z) < -\alpha\}$) can be used in order to guarantee a good convergence velocity and a good damping of the estimation error. The following theorem ensures that the eigenvalues of the matrix $A_{\text{obs}}(t)$ lie in the proposed region $S(\alpha, r)$.

Theorem 3: *Consider the decoupled multiple model (6) and the observer (9). The eigenvalues of the matrix $A_{\text{obs}}(t)$ are assigned inside the prescribed region $S(\alpha, r)$ if there exists a symmetric and positive definite matrix P and a matrix G such that:*

$$\begin{bmatrix} -rP & P\tilde{A} - G\tilde{C}_i \\ \tilde{A}^T P - (G\tilde{C}_i)^T & -rP \end{bmatrix} < 0$$

$$(\tilde{A} + \alpha I)^T P + P(\tilde{A} + \alpha I) - (G\tilde{C}_i)^T - G\tilde{C}_i < 0$$

for $i = 1...L$. The observer gain is given by $\tilde{K} = P^{-1}G$.

Proof: The proof of this theorem is performed by using the notion of *D-stability* of a matrix introduced by Chilali and Gahinet (1996). A matrix A is called *D-stable* if and only if its eigenvalues are inside a region D in the complex plan.

For example, the matrix A is *D-stable* in the region defined by a disk (centred at the origin, with radius r) if the following LMI is satisfied:

$$\begin{bmatrix} -rP & PA \\ A^T P & -rP \end{bmatrix} < 0, \quad P = P^T > 0 \tag{30}$$

Hence, the first inequality of the Theorem 3, that ensures the *D*-stability in the disk, is obtained by substituting matrix A by $A_{\text{obs}}(t)$ in (30), by using the weighting functions properties (2) and finally, by employing the following change of variables $G = P\tilde{K}$.

The second inequality of the Theorem 3, that ensures the *D*-stability in the shifted left-half plane, is given directly by the Theorem 2. Finally, the intersection of these two regions is obtained by finding a common solution to these two inequalities.

5   Simulation example

A simple simulation example is presented in this section in order to illustrate the state estimation of a multiple model and its application to sensor fault detection.

5.1   State estimation of the multiple model

Consider the non-linear system represented by a multiple model with $L = 2$ submodels and where the matrices A_i, B_i and C_i are given by:

$$A_1 = \begin{bmatrix} -2 & 1 & 0.8 \\ 0.6 & -3 & 0 \\ 0.5 & 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 1 & 2 & 1.5 \\ 2 & -1 & -5 & -2 \\ 0.5 & 3 & 1 & -1 \\ -3 & 2.5 & 1 & -0.5 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0.2 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.5 & 1 & 0.25 & 0.75 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0 & 1 & 0.5 \\ 0.2 & 0 & 0.7 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.2 & 0 & 0.9 & 1 \\ 1 & 0.6 & 1 & 0 \\ 0 & 0.2 & 0 & 0.7 \end{bmatrix}$$

The input $u(t)$ is a filtered sequence of piecewise constant signals with variable amplitude $\in [0, 1]$. The weighting functions are obtained from normalised Gaussian functions:

$$\mu_i(\xi(t)) = \frac{\omega_i(\xi(t))}{\sum_{j=1}^{L} \omega_j(\xi(t))}, \quad i = 1...L \tag{31}$$

$$\omega_i(\xi(t)) = \exp\left(\frac{-(\xi(t) - c_i)^2}{\sigma^2}\right) \tag{32}$$

with the standard deviation $\sigma = 0.4$ and the centres $c_1 = 0.3$ and $c_2 = 0.7$. The decision variable $\xi(t)$ of the weighting functions is the input signal $u(t)$.

The evolution of the input signal $u(t)$ and the weighting function $\mu_1(t)$ are shown in Figure 3. As it is clearly shown, the two submodels are requested at any time in order to generate the output of the multiple model. Indeed, the weighting function $\mu_1(t)$ is never null or equal to one.

Eigenvalues of the matrix $\tilde{A}$ are in the left-half complex plane:

$$\lambda = \begin{bmatrix} -1.03 \\ -3.23 \\ -2.73 \\ -1.16 + 4.09i \\ -1.16 - 4.09i \\ -0.58 + 1.33i \\ -0.58 - 1.33i \end{bmatrix} \tag{33}$$

therefore, the multiple model is stable.

A solution of LMIs, in order to obtain the observer gain, can be performed thanks to dedicated numerical toolbox (included in scientific softwares such as Matlab and Scilab). Here, we have used the TKLMITOOL toolbox, developed by El Ghaoui et al. (1997), to solve the LMIs problems.

**Figure 3** Input $u(t)$ and weighting function $\mu_1(t)$
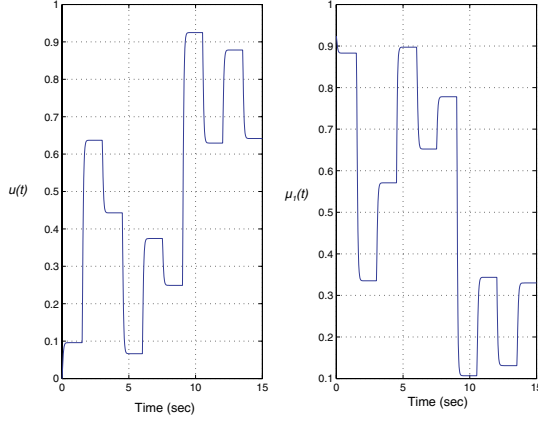
A solution of the theorem 3 is given by:

$$\tilde{K} = \begin{bmatrix} -0.4948 & 0.6717 & 0.3961 \\ -0.2948 & 0.0926 & 0.5942 \\ 0.6912 & -0.8356 & 0.6917 \\ 0.4237 & -0.4415 & 1.3269 \\ 0.4239 & -2.8102 & 1.4686 \\ -0.5212 & 1.6465 & -0.0572 \\ 1.1962 & -4.2020 & 3.2268 \end{bmatrix}$$

with a decay rate $\alpha = 0.55$ and a radius $r = 4.3$.

Figures 4 and 5 show the evolution of the estimation error, where the initial state of the multiple model is equal to:

$$x(0) = \begin{bmatrix} 1 & 0.8 & 0 & 0.2 & 0.7 & 1 & 0.3 \end{bmatrix}^T$$

Figure 4 Estimation error of the submodel 1
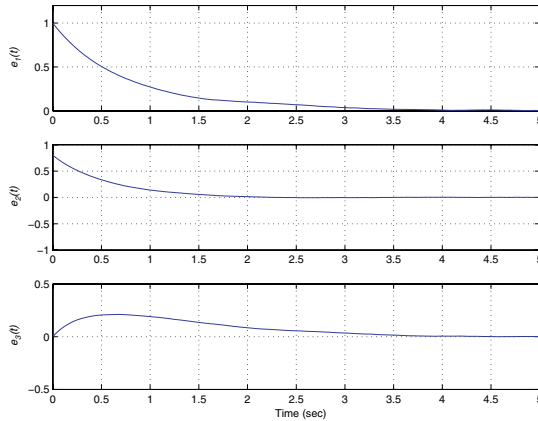
The proposed observer yields a good state estimation of the multiple model. The difference between the initial condition of the multiple model and the multiple observer only provides an error around the time origin.

A noise (normally distributed with zero mean and an unitary variance) is added on the measured output $y(t)$ in order to illustrate the robustness of the proposed observer. As seen from Figure 6, the proposed observer provides a good output estimation even if the used measured signal presents a sensor noise.

Figure 5 Estimation error of the submodel 2
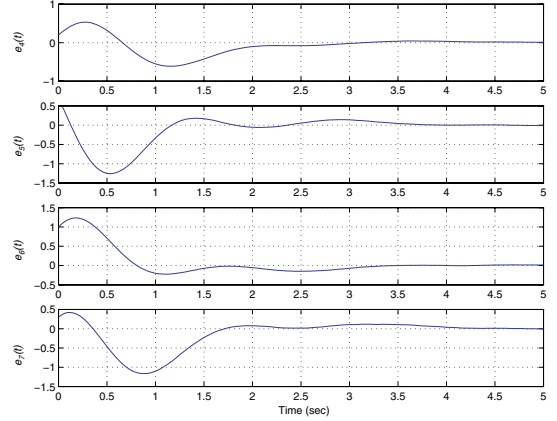
Figure 6 Measured noisy outputs (solid line) and estimated outputs (dashed line)
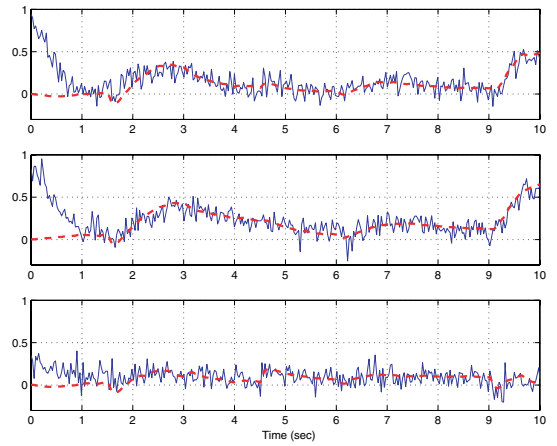
5.2 Sensor fault detection

The proposed observer can be useful with the view of doing a sensor fault detection of a non-linear system represented by a multiple model.

The 'measured' output $\tilde{y}(t)$ is given by:

$$\tilde{y}(t) = y(t) + \delta(t) + \epsilon(t) \tag{34}$$

where $\delta(t)$ is the fault vector and $\epsilon(t)$ the noise (here, normally distributed with zero mean and an unitary variance).

Several fault detection methods based on the state estimation allow the sensor fault detection and isolation (see, e.g. Frank, 1990; Isermann, 2005, for further details about these methods). Classic observer schemes (e.g. Dedicated Observer Scheme (DOS) or Generalised Observer Scheme (GOS)) can be employed in order to generate faults accentuated signals also called *residual signals*.

Assuming that the considered system is undisturbed, a DOS can be employed. In this scheme several observers for a single output are used. The ith observer is only driven by the ith output (observer 1 is driven by $\tilde{y}_1$, observer 2 by $\tilde{y}_2$ and so on). A bank of observers is employed in order to generate the residual signals $r_{i,j}$ sensitive to sensor faults $\delta(t)$. The residual signal $r_{i,j}$ is performed as the difference between the ith measured output of the system and the ith estimated output obtained with the observer j. It allows, coupled with a decision logic, the detection of sensor faults.

An incidence matrix (Table 1) is built in order to analyse how the residual signals $r_{i,j}$ are affected by a fault δ_i. In this matrix, a '1' element indicates that the residual signal $r_{i,j}$ is sensitive to the fault δ_i while a '0' element indicates that the residual signal $r_{i,j}$ does not respond to the fault δ_i. A '?' element indicates that no decision can be taken.

Table 1   Incidence matrix

	δ_1	δ_2	δ_3
$r_{1,1}$	?	0	0
$r_{2,1}$	?	1	0
$r_{3,1}$	?	0	1
$r_{1,2}$	1	?	0
$r_{2,2}$	0	?	0
$r_{3,2}$	0	?	1
$r_{1,3}$	1	0	?
$r_{2,3}$	0	1	?
$r_{3,3}$	0	0	?

The first column of the above incidence matrix is built by considering the following scenario:

1 The output $\tilde{y}_1$ is corrupted by a sensor fault δ_1 but the outputs $\tilde{y}_2$ and $\tilde{y}_3$ are free of fault. Hence, the state estimation performed by the observer 1 is corrupted and the residual signal $r_{i,1}$ can be different from zero. However, compensation phenomenons can appear (due to non-linearity of the observer) and consequently faults can be masked (the value of residual signals $r_{i,1}$ may be null).

2 On the other hand, the state estimation performed by the observers 2 and 3 is correctly performed because these observers are driven by the outputs free of faults. Therefore, the residual signals $r_{1,2}$ and $r_{1,3}$ are undoubtedly sensitive to a fault δ_1 whereas $r_{2,2}$, $r_{3,2}$, $r_{2,3}$ and $r_{3,3}$ are not sensitive to this same fault. Hence, it is this configuration that must be exploited in order to conclude about the presence of a fault on the output $\tilde{y}_1$.

The second and third columns of the incidence matrix can be built in a similar way.

For the considered example, a fault on the output $y_1(t)$ occurs between $t = 3$ and $t = 4.5$ and another fault on the output $y_2(t)$ appears between $t = 7.5$ and $t = 9$. In both cases the faults are biases of constant amplitude equal to one.

Sensor faults are effectively detected using the residual signals (see Figure 7) generated by the bank of observers and the incidence matrix (Table 1). In this simple example, the proposed observer allows the detection and isolation of sensor faults of a decoupled multiple model.

6 Conclusion and perspectives

A new observer, based on a decoupled multiple model, has been used in order to achieve the state estimation of a non-linear system. This observer is an attractive alternative to the well known observer based on a Takagi-Sugeno multiple

model. Indeed, in the proposed architecture the submodels do not share a common state vector and consequently the dimension of the submodels can be different.

Figure 7   Residual signals
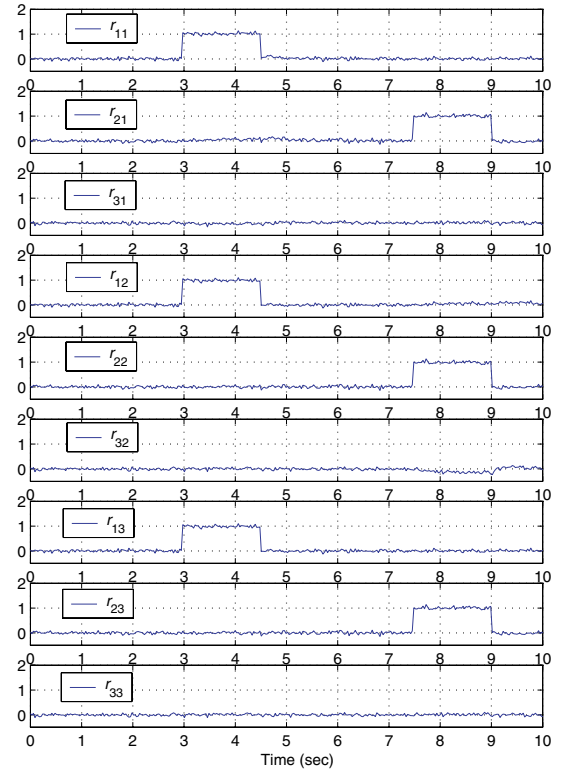
The proposed state estimation has been performed via an extension of the classic proportional gain observer. *Sufficient conditions* for ensuring the convergence of the estimation error have been established in LMIs terms. Dynamic performances of the proposed observer have been enforced thanks to an eigenvalues assignment of the observer in a specific region of the complex plane.

A simulation example allowed to illustrate the state estimation of a multiple model on the one hand and the sensor fault detection and localisation using a bank of observers on the other hand. Encouraging results are obtained in this direction.

Further studies should be held to extend the proposed approach to the design of other types of observers such as proportional integral observer or unknown input observer, in order to take into account possible uncertainties of the multiple model. In order to reduce the conservatism of the proposed solution, other Lyapunov functions, such as *piecewise Lyapunov functions*, could also be considered.

References

Akhenak, A., Chadli, M., Ragot, J. and Maquin, D. (2004) 'Estimation of state and unknown inputs of a nonlinear system represented by a multiple model', *11th IFAC Symposium on Automation in Mining, Mineral and Metal processing*, Nancy, France.

Babuska, R. and Verbruggen, H. (2003) 'Neuro-fuzzy methods for nonlinear system identification', *Annual Reviews in Control*, Vol. 27, No. 1, pp.73–85.

Bergstern, P., Palm, R. and Driankov, D. (2002) 'Observers for Takagi-Sugeno fuzzy systems', *IEEE Transactions on Systems, Man, and Cibernetics*, Vol. 32, No. 1, pp.114–121.

Boyd, S., El Ghaoui, L., Feron, E. and Balakrishnan, V. (1994) *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics 15.

Chadli, M., Maquin, D. and Ragot, J. (2005) 'Stability analysis and design for continuous-time Takagi-Sugeno control systems', *International Journal of Fuzzy Systems*, Vol. 7, No. 3, pp.101–109.

Chilali, M. and Gahinet, P. (1996) 'H-infinity design with pole placement constraints: an LMI approach', *IEEE Transactions on Automatic and Control*, Vol. 41, No. 3, pp.358–367.

El Ghaoui, L., Commeau, J.L. and Chorier, M. (1997) *TKLMITOOL: A Graphical User Interface (GUI) of LMITOOL*, ENSTA, Available at: Ftp://ftp.ensta.fr/pub/elghaoui/, computer software.

Filev, D. (1991) 'Fuzzy modeling of complex systems', *International Journal of Approximate Reasoning*, Vol. 5, No. 3, pp.281–290.

Frank, P. (1990) 'Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy – a survey and some new results', *Automatica*, Vol. 26, No. 3, pp.459–474.

Fritzke, B. (1997) 'Incremental neuro-fuzzy systems', *Applications of Soft Computing, SPIE International Symposium on Optical Science, Engineering and Instrumentation*, San Diego, USA.

Gasso, K., Mourot, G. and Ragot, J. (2001) 'Structure identification in multiple model representation: elimination and merging of local models', *40th IEEE Conference on Decision and Control*, pp.2992–2997, Orlando, USA.

Gatzke, E.P. and Doyle III, F.J. (1999) 'Multiple model approach for CSTR control', *14th IFAC World Congress*, pp.343–348, Beijing, P.R. China.

Gawthrop, P.J. (1995) 'Continuous-time local state local model networks', *1995 IEEE Conference on Systems, Man & Cybernetics*, pp.852–857, Vancouver, British Columbia.

Gregorcic, G. and Lightbody, G. (2000) 'Control of highly nonlinear processes using self-tuning control and multiple/local model approaches', *2000 IEEE International Conference on Intelligent Engineering Systems, INES 2000*, pp.167–171.

Guerra, T.M., Kruszewski, A., Vermeiren, L. and Tirmant, H. (2006) 'Conditions of output stabilization for nonlinear models in the Takagi-Sugeno's form', *Fuzzy Sets and Systems*, Vol. 157, No. 9, pp.1248–1259.

Isermann, R. (2005) 'Model-based fault-detection and diagnosis – status and applications', *Annual Reviews in Control*, Vol. 29, No. 1, pp.71–85.

Kanev, S. and Verhaegen, M. (2006) 'Multiple model weight estimation for models with no common state', *6th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes, SAFEPROCESS*, pp.637–642, Beijing, PR China.

Leith, D.J. and Leithead, W.E. (2000) 'Survey of gain-scheduling analysis and design', *International Journal of Control*, Vol. 73, No. 11, pp.1001–1025.

Lopez-Toribio, C.J., Patton, R.J. and Daley, S. (2000) 'Takagi-Sugeno fuzzy fault-tolerant control of an induction motor', *Neural Computing and Applications*, Vol. 9, No. 1, pp.19–28.

Murray-Smith, R. and Johansen, T.A. (1997) *Multiple Model Approaches to Modelling and Control*, London: Taylor and Francis.

Orjuela, R., Maquin, D. and Ragot, J. (2006) 'Nonlinear system identification using uncoupled state multiple-model approach', *Workshop on Advanced Control and Diagnosis, ACD'2006*, Nancy, France.

Rhee, B. and Won, S. (2006) 'A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design', *Fuzzy Sets and Systems*, Vol. 157, No. 9, pp.1211–1228.

Rodrigues, M., Theilliol, D., Adam-Medina, M. and Sauter, D. (2006) 'A fault detection and isolation scheme for industrial systems based on multiple operating models', *Control Engineering Practice*, doi:10.1016/j.conengprac.2006.02.020.

Shamma, J.S. and Athans, M. (1991) 'Guaranteed properties of gain sheduled control for linear parameter-varying plants', *Automatica*, Vol. 27, No. 3, pp.559–564.

Sontag, E. (1981) 'Nonlinear regulation: the piecewise linear approach', *IEEE Transactions on Automatic Control*, Vol. 26, No. 2, pp.346–358.

Tanaka, K. and Sugeno, M. (1990) 'Stability analysis of fuzzy systems using Lyapunov's direct method', *International Conference of the North American Fuzzy Information Processing Society, NAFIPS*, pp.133–136, Toronto, Canada.

Tanaka, K. and Sugeno, M. (1992) 'Stability analysis and desing of fuzzy control systems', *Fuzzy Sets and Systems*, Vol. 45, No. 2, pp.135–156.

Thiaw, L., Madani, K., Malti, R. and Sow, G. (2007) 'Implementation of recurrent multi-models for system identification', *4th International Conference on Informatics in Control, Automation and Robotics, ICINCO*, pp.314–321, Angers, France.

Ting, C. (2006) 'Stability analysis and design of Takagi-Sugeno fuzzy systems', *Information Sciences*, Vol. 176, No. 19, pp.2817–2845.

Venkat, A.N., Vijaysai, P. and Gudi, R.D. (2003) 'Identification of complex nonlinear processes based on fuzzy decomposition of the steady state space', *Journal of Process Control*, Vol. 13, No. 6, pp.473–488.