Design of observers for Takagi-Sugeno descriptor systems with unknown inputs and application to fault diagnosis

B. MARX[a], D. KOENIG[b] and J. RAGOT[a]

[a] Centre de Recherche en Automatique de Nancy

UMR 7039 CNRS - Nancy Université

2 avenue de la forêt de Haye, 54516 Vandoeuvre-lès-Nancy Cedex, FRANCE

e-mail: {benoit.marx,jose.ragot}@ensem.inpl-nancy.fr

[b] Laboratoire d'Automatique de Grenoble

UMR 5528 CNRS-INPG-UJF

BP 46, 38402 Saint Martin d'Hères, Cedex, FRANCE

e-mail: damien.koenig@inpg.fr

Abstract

This paper presents a method for state-estimation of Takagi-Sugeno descriptor systems (TSDS) affected by unknown inputs (UI). For ease of implementation's sake, the proposed observers are not in descriptor form but in usual form. Sufficient existence conditions of the unknown input observers are given and strict linear matrix inequalities (LMI) are solved to determine the gain of the observers. If the perfect unknown input decoupling is not possible, the UI observer is designed in order to minimise the $\mathcal{L}_2$-gain from the UI to the state estimation error. The two previous objectives can be mixed in order to decouple the estimation to a subset of the UI, while attenuating the $\mathcal{L}_2$ gain from the other UI to the estimation. The proposed UI observers are used for robust fault diagnosis. Fault diagnosis for TSDS is performed by designing a bank of observers. A simple decision logic and thresholds setting allow to determine the occurring fault. The results are established for both the continuous and the discrete time cases. The proposed method is illustrated by a numerical example.

Index Terms

Takagi-Sugeno systems, singular systems, state estimation, unknown input observers, fault diagnosis.

I. INTRODUCTION

The Takagi-Sugeno (TS) model proposed by [14] is a well-known structure to represent nonlinear systems into several linear fuzzy models. In the last two decades, the control and the observation of TS systems have become challenging problems that received a considerable amount of attention. In [19], stability analysis and controller design are addressed, solutions are derived in the linear matrix inequality (LMI) formalism. Relaxed sufficient conditions for fuzzy controllers and fuzzy observers are proposed in [15], and in [16] via a multiple Lyapunov function approach.

The descriptor formalism is very attractive for system modelling, as pointed out in [4], since it describes a wider class of systems including physical systems with non dynamic constraints (e.g. algebraic relations induced in interconnected systems such as power transfer networks or water distribution networks) or jump behaviour. The enhancement of the modelling ability is due to the structure of the dynamic equation which encompasses not only dynamic equations, but also algebraic relations.

Since both TS and descriptor formalisms are attractive in the field of modelling, the TS representation has been generalised to descriptor systems. The stability and the design of state-feedback controllers for TS descriptor systems (TSDS) are characterised via LMI in [17],[18], in particular, the problem of nonlinear model following is treated in [18]. Robust output feedback, and H_∞-control are considered for TSDS in [12] and [21] respectively. The study of TSDS is envisaged with interval methods in [20], in order to take into account the different operating points. Unfortunately, the problem of observer design, and especially the design of unknown input observers, has resulted in very few works.

The design of unknown input observer (UIO) is a crucial problem since, in many practical cases, all input signals cannot be known. Moreover, this class of observers is widely used in the area of fault diagnosis, even if all the inputs are known (see chap. 3 in [13]). The design of UIO has received considerable attention in the case of usual (in opposition to descriptor) linear systems [5], descriptor systems [7], [8], [11] or TS systems [1]. Unfortunately, to the authors' knowledge, the design of UIO has not been treated in the generic case of TSDS. The aim of this paper is not only to generalise the existing works on UIO design to TSDS, but also to apply this new observer in the field of fault diagnosis of TSDS which has not been treated so far.

This paper gives a simple extension to TSDS of the design of observers for the state estimation in the presence of unknown inputs (UI). Under some sufficient conditions, the design of the observer is reduced to the determination of a matrix. The choice of this parameter is performed by solving strict LMIs. If the estimation error cannot be decoupled from the UI, an $\mathcal{L}_2$ observer is proposed to minimise the influence of the UI on the state estimation. The two design objectives can be mixed by decoupling the state estimation from a subset of the UI, and minimising the $\mathcal{L}_2$-gain between the other UI and the state estimation error. The designed observers are used for fault diagnosis, since the UI can encompass the faults and the disturbances affecting the system. Designing several observers attenuating the disturbance effect, and decoupling the estimation from all faults but one lead to the well-known Generalised Observer Scheme (GOS) for fault diagnosis [13]. The design of observers is detailed both in the continuous time case, and in the discrete time case.

The paper is organised as follows: the class of studied systems is defined in section II and the main results about UIO design are detailed in section III. Firstly, the definition of the UIO and the sufficient existence condition are established. Secondly, the computation of the gains of the observer is established. The design of $\mathcal{L}_2$ observers is treated in section IV. Section V deals with the design of observers for both UI decoupling and disturbance attenuation. The application to fault diagnosis is studied in section VI. Section VII is devoted to a numerical example.

Notation 1: In the paper, $\otimes$ denotes the Kronecker product. For a given matrix $\mathbf{X}$, $\mathbf{X}^T$ is the transpose of $\mathbf{X}$, $\mathbf{X} > 0$ (resp. $\mathbf{X} < 0$) means that $\mathbf{X}$ is positive (resp. negative) definite, $\mathbf{X}^+$ denotes the pseudo-inverse of the matrix $\mathbf{X}$, and where $\mathbf{X}^\perp$ is defined by $\mathbf{X}^\perp = \mathbf{I} - \mathbf{X}\mathbf{X}^+$.

II. TAKAGI-SUGENO DESCRIPTOR SYSTEMS

To begin with, the class of systems considered in the present paper is described. In the continuous time case, a TSDS is defined by

$$\mathbf{E}\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t)) \left(\mathbf{A}_i \boldsymbol{x}(t) + \mathbf{B}_i \boldsymbol{u}(t) + \mathbf{D}_i \boldsymbol{d}(t)\right) \tag{1}$$

$$\boldsymbol{y}(t) = \mathbf{C}\boldsymbol{x}(t) + \mathbf{G}\boldsymbol{d}(t) \tag{2}$$

In the discrete time case, a TSDS is defined by

$$\mathbf{E}\boldsymbol{x}_{k+1} = \sum_{i=1}^{r} h_i(\boldsymbol{w}_k) \left(\mathbf{A}_i \boldsymbol{x}_k + \mathbf{B}_i \boldsymbol{u}_k + \mathbf{D}_i \boldsymbol{d}_k\right) \tag{3}$$

$$\boldsymbol{y}_k = \mathbf{C}\boldsymbol{x}_k + \mathbf{G}\boldsymbol{d}_k \tag{4}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the control input, $\boldsymbol{d} \in \mathbb{R}^q$ is the unknown input (disturbance, actuator noise, or hidden message in the recovering framework), and $\boldsymbol{y} \in \mathbb{R}^m$ is the measured output. The matrices $\mathbf{E}$, $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{D}_i$, $\mathbf{C}$ and $\mathbf{G}$ are supposed to be real, known, constant and with appropriate dimensions according to the definition of the signals. The matrix $\mathbf{E}$ may be singular. The activating functions, denoted $h_i(\boldsymbol{w}(t))$, for $i = 1, \ldots, r$, are normalised, and satisfy the following constraints

$$0 \leq h_i(\boldsymbol{w}(t)) \leq 1, \ \sum_{i=1}^{r} h_i(\boldsymbol{w}(t)) = 1, \ \forall t$$

$$0 \leq h_i(\boldsymbol{w}_k) \leq 1, \ \sum_{i=1}^{r} h_i(\boldsymbol{w}_k) = 1, \ \forall k$$

The decision variable $\boldsymbol{w}(t)$ (or $\boldsymbol{w}_k$) is supposed to be real-time accessible, depending on the control input, or on the measured output. Assuming that the same matrix $\mathbf{E}$ appears in all the different sub-models is not restrictive if we consider that the structure of the differential or algebraic relations is imposed by the physical structure of the system, which generally does not change with time. This formalism still encompasses the varying parameters or the nonlinearities since the matrices $\mathbf{A}_i$ are different one from another, and since the activating functions introduce the nonlinear dynamics. An analogous argument justifies the single nature of the output matrix $\mathbf{C}$. The available measurements are determined by the location and the nature of the sensors which generally do not change (the sensors are not removed during the operating time).

III. DESIGN OF UNKNOWN INPUT DECOUPLING OBSERVERS

In this section, our aim is to design a multiple unknown input observer. The UIO are widely used in the field of fault detection and isolation for dynamic systems, because the fault signals are generally unknown. Moreover, a measured signal can be considered as unknown in order to isolate the default corrupting this particular signal (see chapter 3 in [3]).

In this study, the proposed observers are not in descriptor form, in order to reduce the implementation complexity. In the continuous-time case, the proposed multiple UIO is defined by

$$\dot{\boldsymbol{z}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t)) \left(\mathbf{N}_i \boldsymbol{z}(t) + \mathbf{M}_i \boldsymbol{u}(t) + \mathbf{L}_i \boldsymbol{y}(t)\right) \tag{5}$$

$$\hat{\boldsymbol{x}}(t) = \boldsymbol{z}(t) + \mathbf{T}_2 \boldsymbol{y}(t) \tag{6}$$

In the discrete-time case, the proposed multiple UIO is defined by

$$\boldsymbol{z}_{k+1} = \sum_{i=1}^{r} h_i(\boldsymbol{w}_k) \left(\mathbf{N}_i \boldsymbol{z}_k + \mathbf{M}_i \boldsymbol{u}_k + \mathbf{L}_i \boldsymbol{y}_k\right) \tag{7}$$

$$\hat{\boldsymbol{x}}_k = \boldsymbol{z}_k + \mathbf{T}_2 \boldsymbol{y}_k \tag{8}$$

The problem of unknown input decoupling observer (UIDO) design is to find the gains of the UIDO (5-6) (resp. (7-8)), namely $\mathbf{N}_i$, $\mathbf{M}_i$, $\mathbf{L}_i$ and $\mathbf{T}_2$, so that the estimated state $\hat{\boldsymbol{x}}$ asymptotically tends to the state of (1-2) (resp. (3-4)). In other words, the objective is that the estimation error defined by $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ (resp. $\boldsymbol{e}_k = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_k$) tends to zero when $t \to \infty$ (resp. when $k \to \infty$), regardless of the unknown input, the control input, and the initial state.

Firstly, a sufficient rank condition for UI decoupling is given in Lemma 1. Secondly, a sufficient LMI condition for the convergence of the continuous-time UIO is given in Lemma 2 (it is extended to the discrete-time case in Corollary 1). Finally, the results are gathered in Theorem 1 and a design algorithm is given.

Lemma 1: There exists a continuous-time (resp. discrete-time) unknown input decoupling observer (5-6) for (1-2) (resp. (7-8) for (3-4)) if the following condition holds

$$\text{rank } \mathbf{X} = \text{ rank} \begin{bmatrix} \mathbf{D}_1 & \dots & \mathbf{D}_r \\ & \mathbf{I}_r \otimes \mathbf{G} & \end{bmatrix} + n + \text{rank } \mathbf{G} \tag{9}$$

Proof: The estimation error $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ is given by

$$\begin{aligned} \boldsymbol{e}(t) &= \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t) \\ &= \boldsymbol{x}(t) - \boldsymbol{z}(t) - \mathbf{T}_2 \mathbf{C} \boldsymbol{x}(t) - \mathbf{T}_2 \mathbf{G} \boldsymbol{d}(t) \end{aligned}$$

Assume that there exist $\mathbf{T}_1$ and $\mathbf{T}_2$ such that, the following equations hold

$$\mathbf{T}_1 \mathbf{E} + \mathbf{T}_2 \mathbf{C} = \mathbf{I}_n \tag{10}$$

$$\mathbf{T}_2 \mathbf{G} = 0 \tag{11}$$

With (10) and (11), the estimation error becomes $\boldsymbol{e}(t) = \mathbf{T}_1\mathbf{E}\boldsymbol{x}(t) - \boldsymbol{z}(t)$. Its time derivative is given by :

$$\begin{aligned}\dot{\boldsymbol{e}}(t) =& \mathbf{T}_1\mathbf{E}\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{z}}(t)\\ =& \sum_{i=1}^{r} h_i(\boldsymbol{w}(t)) \left[\mathbf{T}_1(\mathbf{A}_i\boldsymbol{x}(t) + \mathbf{B}_i\boldsymbol{u}(t) + \mathbf{D}_i\boldsymbol{d}(t))\right.\\ & \left.-\mathbf{N}_i\boldsymbol{z}(t) - \mathbf{M}_i\boldsymbol{u}(t) - \mathbf{L}_i\boldsymbol{y}(t)\right]\\ =& \sum_{i=1}^{r} h_i(\boldsymbol{w}(t)) \left[\mathbf{N}_i\boldsymbol{e}(t) + (\mathbf{T}_1\mathbf{A}_i - \mathbf{N}_i\mathbf{T}_1\mathbf{E} - \mathbf{L}_i\mathbf{C})\boldsymbol{x}(t)\right.\\ & \left.+(\mathbf{T}_1\mathbf{B}_i - \mathbf{M}_i)\boldsymbol{u}(t) + (\mathbf{T}_1\mathbf{D}_i - \mathbf{L}_i\mathbf{G})\boldsymbol{d}(t)\right]\end{aligned} \tag{12}$$

The time derivative of the estimation error is given by :

$$\dot{\boldsymbol{e}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t))\mathbf{N}_i\boldsymbol{e}(t) \tag{13}$$

if the following constraints hold for $i = 1, \ldots, r$

$$\mathbf{I}_n = \mathbf{T}_1\mathbf{E} + \mathbf{T}_2\mathbf{C} \tag{14}$$

$$\mathbf{0} = \mathbf{T}_2\mathbf{G} \tag{15}$$

$$\mathbf{0} = \mathbf{T}_1\mathbf{A}_i - \mathbf{N}_i\mathbf{T}_1\mathbf{E} - \mathbf{L}_i\mathbf{C} \tag{16}$$

$$\mathbf{0} = \mathbf{T}_1\mathbf{B}_i - \mathbf{M}_i \tag{17}$$

$$\mathbf{0} = \mathbf{T}_1\mathbf{D}_i - \mathbf{L}_i\mathbf{G} \tag{18}$$

In order to find the gains of the UIDO, according to the constraints (14-18), new parameters $\mathbf{K}_i = \mathbf{N}_i\mathbf{T}_2 - \mathbf{L}_i$ are introduced in (16). Then, the UIDO exists if, for $i = 1, \ldots, r$, the following statements are true

$$\mathbf{N}_i = \mathbf{T}_1\mathbf{A}_i + \mathbf{K}_i\mathbf{C} \tag{19}$$

$$\mathbf{I}_n = \mathbf{T}_1\mathbf{E} + \mathbf{T}_2\mathbf{C} \tag{20}$$

$$\mathbf{0} = \mathbf{T}_2\mathbf{G} \tag{21}$$

$$\mathbf{0} = \mathbf{T}_1\mathbf{D}_i + \mathbf{K}_i\mathbf{G} \tag{22}$$

$$\mathbf{M}_i = \mathbf{T}_1\mathbf{B}_i \tag{23}$$

$$\mathbf{L}_i = \mathbf{N}_i\mathbf{T}_2 - \mathbf{K}_i \tag{24}$$

Verifying the constraints (19-22) reduces to finding $\mathbf{\Theta} \in \mathbb{R}^{n\times(n+m(r+1))}$ such that

$$\mathbf{\Theta}\mathbf{X} = \mathbf{Y} \tag{25}$$

$$\mathbf{N}_i = \mathbf{\Theta}\mathbf{Y}_i \tag{26}$$

where $\mathbf{\Theta}$, is given by

$$\mathbf{\Theta} = \left[\begin{array}{cc|cccc} \mathbf{T}_1 & \mathbf{T}_2 & \mathbf{K}_1 & \mathbf{K}_2 & \ldots & \mathbf{K}_r \end{array}\right] \tag{27}$$

Once $\mathbf{\Theta}$ is known, $\mathbf{M}_i$ and $\mathbf{L}_i$ are deduced from (23) and (24) respectively. The equation (25) is solvable in the variable $\mathbf{\Theta}$ if the following condition holds

$$\text{rank}\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix} = \text{rank } \mathbf{X} \tag{28}$$

where the matrices $\mathbf{X} \in \mathbb{R}^{(n+m(r+1))\times(n+q(r+1))}$ and $\mathbf{Y} \in \mathbb{R}^{n\times(n+q(r+1))}$ are defined by

$$\mathbf{X} = \left[\begin{array}{cc|c}\mathbf{E} & \mathbf{0}_{n\times q} & \mathbf{D}_1 \ \dots \ \mathbf{D}_r \\ \mathbf{C} & \mathbf{G} & \mathbf{0}_{m\times q} \ \dots \ \mathbf{0}_{m\times q} \\ \hline \mathbf{0}_{rm\times n} & \mathbf{0}_{rm\times q} & \mathbf{I}_r \otimes \mathbf{G}\end{array}\right] \tag{29}$$

$$\mathbf{Y} = \left[\begin{array}{cc|c}\mathbf{I}_n & \mathbf{0}_{n\times q} & \mathbf{0}_{n\times rq}\end{array}\right] \tag{30}$$

Obviously, with (30) and (29), the condition (28) becomes

$$\text{rank}\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix} = n + \text{rank } \mathbf{G} + \text{rank}\begin{bmatrix}\mathbf{D}_1 \ \dots \ \mathbf{D}_r \\ \mathbf{I}_r \otimes \mathbf{G}\end{bmatrix}$$

$$\text{rank } \mathbf{X} = \text{rank}\left[\begin{array}{cc|c}\mathbf{E} & \mathbf{0} & \mathbf{D}_1 \dots \mathbf{D}_r \\ \mathbf{C} & \mathbf{G} & \mathbf{0} \ \dots \ \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_r \otimes \mathbf{G}\end{array}\right]$$

Then (28) is equivalent to (9). In the discrete-time case, the proof is very similar, thus it is omitted. ■

Lemma 2: The estimation error of the UIO (5-6) for (1-2) tends to zero if there exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ and a matrix $\underline{\mathbf{Z}} \in \mathbb{R}^{n\times(n+m(r+1))}$ verifying the following LMI for $i = 1, \dots, r$

$$(\mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i)^T\mathbf{P} + \mathbf{P}\mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i + (\mathbf{X}^{\perp}\mathbf{Y}_i)^T\underline{\mathbf{Z}}^T + \underline{\mathbf{Z}}\mathbf{X}^{\perp}\mathbf{Y}_i < 0 \tag{31}$$

where $\otimes$ is the Kronecker product. The matrices $\mathbf{X}$ and $\mathbf{Y}$ are defined by (29) and (30) respectively, and $\mathbf{Y}_i \in \mathbb{R}^{(n+m(r+1))\times n}$ are defined by

$$\mathbf{Y}_i = \left[\begin{array}{c}\mathbf{A}_i \\ \mathbf{0}_{m\times n} \\ \hline \boldsymbol{e}_i \otimes \mathbf{C}\end{array}\right]$$

where $\boldsymbol{e}_i \in \mathbb{R}^{r\times 1}$ is the column vector with all its components equal to 0, except the i^{th} equal to 1.

Proof: Suppose that (9) is satisfied, then (25) is solvable and the solutions $\mathbf{\Theta}$ are given by

$$\mathbf{\Theta} = \mathbf{Y}\mathbf{X}^{+} + \mathbf{Z}\mathbf{X}^{\perp} \tag{32}$$

where $\mathbf{Z} \in \mathbb{R}^{n\times(n+m(r+1))}$ is an arbitrary matrix.

With (26) and (32), the matrices $\mathbf{N}_i$ are defined by $\mathbf{N}_i = \mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i + \mathbf{Z}\mathbf{X}^{\perp}\mathbf{Y}_i$. The state estimation error tends to zero if the polytopic system (13) is stable. A well known stability condition for polytopic system (see [2]) is the existence of a symmetric positive definite matrix $\mathbf{P}$ verifying $\mathbf{N}_i^T\mathbf{P} + \mathbf{P}\mathbf{N}_i < 0$ for $i = 1, \dots, r$. Then the UIDO provides an estimate of the system state if there exists a matrix $\mathbf{Z}$ such that $\mathbf{P}(\mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i + \mathbf{Z}\mathbf{X}^{\perp}\mathbf{Y}_i) + (\mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i + \mathbf{Z}\mathbf{X}^{\perp}\mathbf{Y}_i)^T\mathbf{P} < 0$, for all $i = 1, \dots, r$. Setting $\underline{\mathbf{Z}} = \mathbf{P}\mathbf{Z}$ then (31) is obtained, which completes the proof. ■

This result is extended to the discrete-time case.

Corollary 1: The estimation error of the UIO (7-8) for (3-4) tends to zero if there exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and a matrix $\underline{\mathbf{Z}} \in \mathbb{R}^{n \times (n+m(r+1))}$ verifying the following LMI for $i = 1, \dots, r$

$$\begin{bmatrix} \mathbf{\Phi}_i & (\mathbf{X}^{\perp}\mathbf{Y}_i)^T \underline{\mathbf{Z}}^T \\ \underline{\mathbf{Z}}(\mathbf{X}^{\perp}\mathbf{Y}_i) & -\mathbf{P} \end{bmatrix} < 0 \tag{33}$$

where $\mathbf{\Phi}_i$ is defined by

$$\begin{aligned} \mathbf{\Phi}_i = {} & (\mathbf{X}^{\perp}\mathbf{Y}_i)^T \underline{\mathbf{Z}}^T (\mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i) + (\mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i)^T \underline{\mathbf{Z}} (\mathbf{X}^{\perp}\mathbf{Y}_i) \\ & + (\mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i)^T \mathbf{P} (\mathbf{Y}\mathbf{X}^{+}\mathbf{Y}_i) - \mathbf{P} \end{aligned}$$

Proof: Proof is similar to the continuous-time case, apart from the condition for the stability of the state estimation error. In the discrete-time case, $\boldsymbol{e}_k$ tends to zeros if there exists a symmetric positive definite matrix $\mathbf{P}$ such that $\mathbf{N}_i^T \mathbf{P} \mathbf{N}_i - \mathbf{P} < 0$ for $i = 1, \dots, r$. With $\mathbf{P}\mathbf{Z} = \underline{\mathbf{Z}}$, the LMI (33) follows. ■

Theorem 1: There exists a continuous-time (resp. discrete-time) UIO (5-6) for (1-2) (resp. (7-8) for (3-4)) if the condition (9) is satisfied and if there exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and a matrix $\underline{\mathbf{Z}} \in \mathbb{R}^{n \times (n+m(r+1))}$ verifying (31) (resp. (33)), for $i = 1, \dots, r$.

Finally, the design of UIO for continuous-time (resp. discrete-time) TSDS is reduced to the following procedure.

Step 1. Verify the existence condition (9).

Step 2. Solve the LMI (31) (resp. (33)) in $\mathbf{P}$ and $\underline{\mathbf{Z}}$.

Step 3. Compute $\mathbf{Z}$ with $\mathbf{Z} = \mathbf{P}^{-1}\underline{\mathbf{Z}}$. For a given $\mathbf{Z}$, $\mathbf{\Theta}$ is deduced from (32), the matrices $\mathbf{N}_i$, $\mathbf{M}_i$ and $\mathbf{L}_i$ are derived from (19), (23) and (24) respectively.

This result unifies the results obtained, on the one hand, in the field of the descriptor systems with UI [6],[9], [10] and, on the other hand, in the field of the TS systems with UI [1]. It is useful because a TSDS cannot be reduced, either to a single singular system (it would not handle the nonlinearities due to the weighting functions h_i), or to a regular TS system (it would not handle the algebraic relation between the state variables). The existence condition (9) can be linked to previous works concerning single descriptor systems [9], [10]. Considering (9) for a single descriptor system, would lead to the condition

$$\text{rank} \begin{bmatrix} \mathbf{E} & \mathbf{D}_1 & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{G} & \mathbf{0} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{G} \end{bmatrix} + n + \text{rank}\ \mathbf{G} \tag{34}$$

One can note that (34) is equivalent to the condition (21) or (31) of [9], and is also equivalent to the condition (A3a) in [10]. Moreover, the present paper gives only sufficient conditions, whereas [9] gave necessary and sufficient conditions. This difference appears because the present paper is basically written for TS systems, thus the weighting functions cause conservatism since the matrix $\sum_{i=1}^{r} h_i(\boldsymbol{w}(t))\mathbf{A}_i$ can take all the possible values in the polytope defined by its vertices A_i.

IV. Design of unknown input attenuating observers

In this section, the aim is to design an observer for TSDS in order to minimise the influence of the UI on the state estimation when the perfect decoupling is not possible. The chosen criterion to minimise is the $\mathcal{L}_2$-gain between the unknown input and the state estimation error. This approach is less restrictive than the design of an UIO since the structural condition (9) is partially relaxed.

As pointed out in the section of UIO design, the estimation error e is governed by a non singular TS system (12), thus in order to bound the $\mathcal{L}_2$-gain from the UI to e, and establish the sufficient conditions of the so-called $\mathcal{L}_2$ observer, the following lemma concerning $\mathcal{L}_2$-gain of TS-systems is needed.

Lemma 3: [2] Consider the continuous-time TS-system defined by

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t))(\mathbf{A}_i\boldsymbol{x}(t) + \mathbf{B}_i\boldsymbol{u}(t)) \tag{35}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t))\mathbf{C}_i\boldsymbol{x}(t) \tag{36}$$

and the discrete-time TS-system defined by

$$\boldsymbol{x}_{k+1} = \sum_{i=1}^{r} h_i(\boldsymbol{w}_k)(\mathbf{A}_i\boldsymbol{x}_k + \mathbf{B}_i\boldsymbol{u}_k) \tag{37}$$

$$\boldsymbol{y}_k = \sum_{i=1}^{r} h_i(\boldsymbol{w}_k)\mathbf{C}_i\boldsymbol{x}_k \tag{38}$$

The system (35-36) (resp. (37-38)) is stable and verifies $||\boldsymbol{y}||_2 < \gamma||\boldsymbol{u}||_2$ if there exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ such that (39) (resp. (40)) is satisfied for $i = 1, \dots, r$.

$$\begin{bmatrix} \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i + \mathbf{C}_i^T\mathbf{C}_i & \mathbf{P}\mathbf{B}_i \\ \mathbf{B}_i^T\mathbf{P} & -\gamma^2\mathbf{I} \end{bmatrix} < 0 \tag{39}$$

$$\begin{bmatrix} \mathbf{A}_i^T\mathbf{P}\mathbf{A}_i + \mathbf{C}_i^T\mathbf{C}_i - \mathbf{P} & \mathbf{A}_i^T\mathbf{P}\mathbf{B}_i \\ \mathbf{B}_i^T\mathbf{P}\mathbf{A}_i & \mathbf{B}_i^T\mathbf{P}\mathbf{B}_i - \gamma^2\mathbf{I} \end{bmatrix} < 0 \tag{40}$$

For a given real positive γ, an observer is said to be an unknown input attenuating observer (UIAO) of $\mathcal{L}_2$-gain γ, if the state estimation error, $\boldsymbol{e}$, and the unknown input, $\boldsymbol{d}$, satisfy $||\boldsymbol{e}||_2 < \gamma||\boldsymbol{d}||_2$.

Theorem 2: There exists an UIAO (5-6), with an $\mathcal{L}_2$-gain lower than γ, for the system (1-2), if the condition (41) is satisfied, and if there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ and matrices $\underline{\mathbf{Z}} \in \mathbb{R}^{n\times(n+m)}$ and $\underline{\mathbf{K}}_i \in \mathbb{R}^{n\times m}$, verifying the LMI (42) for $i = 1, \dots, r$.

$$\text{rank} \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{C} & \mathbf{G} \end{bmatrix} = n + \text{rank } \mathbf{G} \tag{41}$$

$$\begin{bmatrix} \mathbf{\Psi}_{i,1} & \mathbf{\Psi}_{i,2} \\ \mathbf{\Psi}_{i,2}^T & -\gamma^2\mathbf{I}_q \end{bmatrix} < 0 \tag{42}$$

where the matrices $\mathbf{\Psi}_{i,1}$, and $\mathbf{\Psi}_{i,2}$ are given by

$$\begin{aligned}\mathbf{\Psi}_{i,1} =& \mathbf{PYX}_1^+\mathbf{A}_i + \underline{\mathbf{Z}}\mathbf{X}_1^\perp\mathbf{A}_i + \underline{\mathbf{K}}_i\mathbf{C} \\ &+ (\mathbf{PYX}_1^+\mathbf{A}_i + \underline{\mathbf{Z}}\mathbf{X}_1^\perp\mathbf{A}_i + \underline{\mathbf{K}}_i\mathbf{C})^T + \mathbf{I}_n \\ \mathbf{\Psi}_{i,2} =& \mathbf{PYX}_1^+\mathbf{D}_i + \underline{\mathbf{Z}}\mathbf{X}_1^\perp\mathbf{D}_i + \underline{\mathbf{K}}_i\mathbf{G}\end{aligned}$$

where $\mathbf{X}_1^+ \in \mathbb{R}^{(n+q)\times n}$, $\mathbf{X}_2^+ \in \mathbb{R}^{(n+q)\times m}$, $\mathbf{X}_1^\perp \in \mathbb{R}^{(n+m)\times n}$ and $\mathbf{X}_2^\perp \in \mathbb{R}^{(n+m)\times m}$ are defined by

$$\begin{bmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{C} & \mathbf{G}\end{bmatrix}^+ = \begin{bmatrix}\mathbf{X}_1^+ & \mathbf{X}_2^+\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{C} & \mathbf{G}\end{bmatrix}\begin{bmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{C} & \mathbf{G}\end{bmatrix}^+ - \mathbf{I}_{n+m} = \begin{bmatrix}\mathbf{X}_1^\perp & \mathbf{X}_2^\perp\end{bmatrix}$$

Proof: If (41) is satisfied, then there exist $\mathbf{T}_1$ and $\mathbf{T}_2$ such that

$$\begin{bmatrix}\mathbf{T}_1 & \mathbf{T}_2\end{bmatrix}\mathbf{X} = \mathbf{Y}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are given by

$$\mathbf{X} = \begin{bmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{C} & \mathbf{G}\end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix}\mathbf{I}_n & \mathbf{0}_{n\times m}\end{bmatrix}$$

and, for any arbitrary matrix $\mathbf{Z}$, $\mathbf{T}_1$ and $\mathbf{T}_2$ are given by

$$\mathbf{T}_1 = \mathbf{YX}_1^+ + \mathbf{ZX}_1^\perp \tag{43}$$

$$\mathbf{T}_2 = \mathbf{YX}_2^+ + \mathbf{ZX}_2^\perp \tag{44}$$

Following the proof of Theorem 1, if (19), (23) and (24) hold, the state estimation error $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ is governed by

$$\dot{\boldsymbol{e}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t))((\mathbf{T}_1\mathbf{A}_i + \mathbf{K}_i\mathbf{C})\boldsymbol{e}(t) + (\mathbf{T}_1\mathbf{D}_i + \mathbf{K}_i\mathbf{G})\boldsymbol{d}(t)) \tag{45}$$

According to Lemma 3, $||\boldsymbol{e}(t)||_2 < \gamma||\boldsymbol{d}(t)||_2$ if there exists a symmetric positive definite matrix $\mathbf{P}$ such that the following LMI hold for $i = 1, \ldots, r$

$$\begin{bmatrix}\mathbf{\Psi}_i + \mathbf{I}_n & \mathbf{PT}_1\mathbf{D}_i + \mathbf{PK}_i\mathbf{G}\\ \mathbf{D}_i^T\mathbf{T}_1^T\mathbf{P} + \mathbf{G}^T\mathbf{K}_i^T\mathbf{P} & -\gamma^2\mathbf{I}_q\end{bmatrix} < 0$$

where $\mathbf{\Psi}_i$ is given by

$$\mathbf{\Psi}_i = (\mathbf{T}_1\mathbf{A}_i + \mathbf{K}_i\mathbf{C})^T\mathbf{P} + \mathbf{P}(\mathbf{T}_1\mathbf{A}_i + \mathbf{K}_i\mathbf{C})$$

With $\underline{\mathbf{K}}_i = \mathbf{PK}_i$ and $\underline{\mathbf{Z}} = \mathbf{PZ}$, the LMI (42) follows, which completes the proof. ■

Remark 1: Obviously, the condition (41) is less restrictive than (9).

Remark 2: γ^2 can be considered as a variable to be minimised during the LMI optimisation, to obtain an optimal UI attenuation.

Corollary 2: There exists an UIAO (7-8) with an$\mathcal{L}_2$-gain lower than a given real positive γ for the system (3-4), if the condition (41) is satisfied, and if there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$, and matrices $\underline{\mathbf{Z}} \in \mathbb{R}^{n\times(n+m)}$ and $\underline{\mathbf{K}}_i \in \mathbb{R}^{n\times m}$, verifying the following LMI for $i = 1, \ldots, r$

$$\begin{bmatrix} \mathbf{I}_n - \mathbf{P} & \mathbf{0} & \mathbf{\Phi}_{i,1}^T \\ \mathbf{0} & -\gamma^2 \mathbf{I}_n & \mathbf{\Phi}_{i,2}^T \\ \mathbf{\Phi}_{i,1} & \mathbf{\Phi}_{i,2} & -\mathbf{P} \end{bmatrix} < 0 \tag{46}$$

where $\mathbf{\Phi}_{i,1}$ and $\mathbf{\Phi}_{i,2}$ are defined by

$$\mathbf{\Phi}_{i,1} = \mathbf{P}\mathbf{Y}\mathbf{X}_1^+\mathbf{A}_i + \underline{\mathbf{Z}}\mathbf{X}_1^\perp\mathbf{A}_i + \underline{\mathbf{K}}_i\mathbf{C}$$

$$\mathbf{\Phi}_{i,2} = \mathbf{P}\mathbf{Y}\mathbf{X}_1^+\mathbf{D}_i + \underline{\mathbf{Z}}\mathbf{X}_1^\perp\mathbf{D}_i + \underline{\mathbf{K}}_i\mathbf{G}$$

Proof: The proof follows the lines of the proof of Theorem 2 with a Schur complement and is therefore omitted. ■

Finally the design of UIAO for continuous-time (resp. discrete-time) TSDS is reduced to the following procedure.

Step 1. Verify the existence condition (41).

Step 2. Solve the LMI (42) (resp. (46)) in $\mathbf{P}$, $\underline{\mathbf{Z}}$ and $\underline{\mathbf{K}}_i$.

Step 3. Compute $\mathbf{Z}$ and $\mathbf{K}_i$ with $\mathbf{Z} = \mathbf{P}^{-1}\underline{\mathbf{Z}}$ and $\mathbf{K}_i = \mathbf{P}^{-1}\underline{\mathbf{K}}_i$ respectively. The matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are obtained by (43) and (44). The matrices $\mathbf{N}_i$, $\mathbf{M}_i$ and $\mathbf{L}_i$ are derived from (19), (23) and (24) respectively.

V. Design of unknown input decoupling and attenuating observer

If the unknown inputs are too numerous or if their distribution structure makes the perfect unknown input decoupling of the estimation impossible (i.e. if the structural condition (9) is not satisfied), a compromise can be made in order to design an observer ensuring two complementary objectives with less restrictive existence conditions. Firstly, the state estimation is perfectly decoupled to a subset of the UI denoted $\boldsymbol{d}(t)$. Secondly, the $\mathcal{L}_2$-gain between the other UI, denoted $\bar{\boldsymbol{d}}(t)$, to the state estimation error is minimised, thus the state estimation is made maximally robust to these UI. Partitioning the UI into $\boldsymbol{d}(t)$ and $\bar{\boldsymbol{d}}(t)$ the system (1-2) can be written as

$$\mathbf{E}\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t))\big(\mathbf{A}_i\boldsymbol{x}(t) + \mathbf{B}_i\boldsymbol{u}(t) + \mathbf{D}_i\boldsymbol{d}(t) + \overline{\mathbf{D}}_i\bar{\boldsymbol{d}}(t)\big) \tag{47}$$

$$\boldsymbol{y}(t) = \mathbf{C}\boldsymbol{x}(t) + \mathbf{G}\boldsymbol{d}(t) + \overline{\mathbf{G}}\,\bar{\boldsymbol{d}}(t) \tag{48}$$

where $\boldsymbol{d}(t) \in \mathbb{R}^q$ and $\bar{\boldsymbol{d}}(t) \in \mathbb{R}^{\bar{q}}$. The partition of the UI into $\boldsymbol{d}(t)$ and $\bar{\boldsymbol{d}}(t)$ is such that the perfect decoupling condition is satisfied for $(\mathbf{E}, \mathbf{C}, \mathbf{G}, \mathbf{D}_1, \ldots, \mathbf{D}_r)$, then the $\mathcal{L}_2$-gain from $\bar{\boldsymbol{d}}(t)$ to the state estimation error is minimised.

The designs of UIDO and UIAO are combined to derive the design of an unknown input decoupling/attenuating observer (UIDAO). The sufficient existence conditions are given in the following theorem.

Theorem 3: There exists an observer (5-6) ensuring perfect decoupling to $\boldsymbol{d}(t)$ and maximally robust to $\bar{\boldsymbol{d}}(t)$ if the condition (49) is satisfied and if there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ and a matrix

$\overline{\mathbf{Z}} \in \mathbb{R}^{n\times(n+(r+1)m)}$ solution of the minimisation of γ under the LMI constraint (50) for $i = 1, \ldots, r$.

$$\text{rank}\ \overline{\mathbf{X}} = n + \text{rank}\begin{bmatrix}\mathbf{G} & \overline{\mathbf{G}}\end{bmatrix} + \text{rank}\begin{bmatrix}\mathbf{D}_1 \ \ldots \ \mathbf{D}_r \\ \mathbf{I}_r \otimes \mathbf{G}\end{bmatrix} \tag{49}$$

$$\begin{bmatrix}\overline{\boldsymbol{\Psi}}_{i,1} & \overline{\boldsymbol{\Psi}}_{i,2} \\ \overline{\boldsymbol{\Psi}}_{i,2}^T & -\gamma^2\mathbf{I}\end{bmatrix} < 0 \tag{50}$$

where $\overline{\boldsymbol{\Psi}}_{i,1}$ and $\overline{\boldsymbol{\Psi}}_{i,2}$ are given by

$$\begin{aligned}\overline{\boldsymbol{\Psi}}_{i,1} =& \mathbf{P}\overline{\mathbf{Y}}\overline{\mathbf{X}}^{+}\mathbf{Y}_i + \overline{\mathbf{Z}}\overline{\mathbf{X}}^{\perp}\mathbf{Y}_i \\ &+ (\overline{\mathbf{Y}}\overline{\mathbf{X}}^{+}\mathbf{Y}_i)^T\mathbf{P} + (\overline{\mathbf{X}}^{\perp}\mathbf{Y}_i)^T\overline{\mathbf{Z}}^T + \mathbf{I}_n \\ \overline{\boldsymbol{\Psi}}_{i,2} =& \mathbf{P}\overline{\mathbf{Y}}\overline{\mathbf{X}}^{+}\overline{\mathbf{Y}}_i + \overline{\mathbf{Z}}\overline{\mathbf{X}}^{\perp}\overline{\mathbf{Y}}_i\end{aligned}$$

where $\overline{\mathbf{X}}$, $\overline{\mathbf{Y}}$, $\mathbf{Y}_i$ and $\overline{\mathbf{Y}}_i$ are given by

$$\overline{\mathbf{X}} = \left[\begin{array}{ccc|c}\mathbf{E} & \mathbf{0}_{n\times q} & \mathbf{0}_{n\times\bar{q}} & \mathbf{D}_1 \ \ldots \ \mathbf{D}_r \\ \mathbf{C} & \mathbf{G} & \overline{\mathbf{G}} & \mathbf{0}_{m\times q} \ \ldots \ \mathbf{0}_{m\times q} \\ \hline \mathbf{0}_{rm\times n} & \mathbf{0}_{rm\times q} & \mathbf{0}_{rm\times\bar{q}} & \mathbf{I}_r \otimes \mathbf{G}\end{array}\right] \tag{51}$$

$$\overline{\mathbf{Y}} = \left[\begin{array}{cc|c}\mathbf{I}_n & \mathbf{0}_{n\times q+\bar{q}} & \mathbf{0}_{n\times rq}\end{array}\right]$$

$$\mathbf{Y}_i = \left[\begin{array}{c}\mathbf{A}_i \\ \mathbf{0}_{m\times n} \\ \hline \boldsymbol{e}_i \otimes \mathbf{C}\end{array}\right] \quad \overline{\mathbf{Y}}_i = \left[\begin{array}{c}\overline{\mathbf{D}}_i \\ \mathbf{0}_{m\times\bar{q}} \\ \hline \boldsymbol{e}_i \otimes \overline{\mathbf{G}}\end{array}\right]$$

Proof: The state estimation error $\boldsymbol{e}(t)$ is governed by the following system

$$\begin{aligned}\dot{\boldsymbol{e}}(t) =& \sum_{i=1}^{r} h_i(\boldsymbol{w}(t))(\mathbf{N}_i\boldsymbol{e}(t) + (\mathbf{T}_1\mathbf{A}_i - \mathbf{N}_i\mathbf{T}_1\mathbf{E} - \mathbf{L}_i\mathbf{C})\boldsymbol{x}(t) \\ &+ (\mathbf{T}_1\mathbf{B}_i - \mathbf{M}_i)\boldsymbol{u}(t) + (\mathbf{T}_1\overline{\mathbf{D}}_i - \mathbf{L}_i\overline{\mathbf{G}})\overline{\boldsymbol{d}}(t) \\ &+ (\mathbf{T}_1\mathbf{D}_i - \mathbf{L}_i\mathbf{G})\boldsymbol{d}(t) + \mathbf{T}_2\mathbf{G}\dot{\boldsymbol{d}}(t) + \mathbf{T}_2\overline{\mathbf{G}}\dot{\overline{\boldsymbol{d}}}(t))\end{aligned} \tag{52}$$

Following a similar argument to that in the proof of Theorem 1, if the observer parameters satisfy the constraints (19-24) and $\mathbf{T}_2\overline{\mathbf{G}} = 0$, $\boldsymbol{e}(t)$ is governed by

$$\dot{\boldsymbol{e}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t))(\mathbf{N}_i\boldsymbol{e}(t) + (\mathbf{T}_1\overline{\mathbf{D}}_i + \mathbf{K}_i\overline{\mathbf{G}})\overline{\boldsymbol{d}}(t)) \tag{53}$$

These constraints can be written as $\boldsymbol{\Theta}\overline{\mathbf{X}} = \overline{\mathbf{Y}}$, with $\boldsymbol{\Theta}$ defined by (27). This equation can be solved if and only if $\text{rank}\begin{bmatrix}\overline{\mathbf{X}}^T & \overline{\mathbf{Y}}^T\end{bmatrix}^T = \text{rank}\ \overline{\mathbf{X}}$ which is equivalent to the condition (49). If condition (49) is satisfied, then, for any arbitrary matrix $\mathbf{Z}$, $\boldsymbol{\Theta}$ is given by

$$\boldsymbol{\Theta} = \overline{\mathbf{Y}}\overline{\mathbf{X}}^{+} + \mathbf{Z}\overline{\mathbf{X}}^{\perp} \tag{54}$$

Then, the only parameter to be found is the matrix $\mathbf{Z}$. Since the matrices $\mathbf{N}_i$ and $(\mathbf{T}_1\overline{\mathbf{D}}_i - \mathbf{L}_i\overline{\mathbf{G}})$ can be written as

$$\mathbf{N}_i = \mathbf{\Theta}\mathbf{Y}_i = \overline{\mathbf{Y}\mathbf{X}}^{+}\mathbf{Y}_i + \mathbf{Z}\overline{\mathbf{X}}^{\perp}\mathbf{Y}_i$$

$$\mathbf{T}_1\overline{\mathbf{D}}_i + \mathbf{K}_i\overline{\mathbf{G}} = \mathbf{\Theta}\overline{\mathbf{Y}}_i = \overline{\mathbf{Y}\mathbf{X}}^{+}\overline{\mathbf{Y}}_i + \mathbf{Z}\overline{\mathbf{X}}^{\perp}\overline{\mathbf{Y}}_i$$

the stability condition of (53) follows the same lines as in the proof of Theorem 2. Rewriting the stability condition (39) for the triplet $(\mathbf{N}_i, (\mathbf{T}_1\overline{\mathbf{D}}_i - \mathbf{K}_i\overline{\mathbf{G}}), \mathbf{I}_n)$, and setting $\mathbf{P}\mathbf{Z} = \overline{\mathbf{Z}}$, the LMI condition (50) follows. ■

Remark 3: Obviously, the condition (49) is less restrictive than (9). To obtain perfect decoupling to all the UI of (47-48), $\mathbf{D}_i$ and $\mathbf{G}$ should be replaced by $\begin{bmatrix}\mathbf{D}_i & \overline{\mathbf{D}}_i\end{bmatrix}$ and $\begin{bmatrix}\mathbf{G} & \overline{\mathbf{G}}\end{bmatrix}$ respectively in (49) and (51), which would lead to a more restrictive existence condition.

A similar result can be given for discrete-time systems defined by

$$\mathbf{E}\boldsymbol{x}_{k+1} = \sum_{i=1}^{r} h_i(\boldsymbol{w}_k)\left(\mathbf{A}_i\boldsymbol{x}_k + \mathbf{B}_i\boldsymbol{u}_k + \mathbf{D}_i\boldsymbol{d}_k + \overline{\mathbf{D}}_i\overline{\boldsymbol{d}}_k\right) \quad (55)$$

$$\boldsymbol{y}_k = \mathbf{C}\boldsymbol{x}_k + \mathbf{G}\boldsymbol{d}_k + \overline{\mathbf{G}}\,\overline{\boldsymbol{d}}_k \quad (56)$$

Corollary 3: There exists an observer (7-8) ensuring perfect decoupling to $\boldsymbol{d}_k$ and maximally robust to $\overline{\boldsymbol{d}}_k$ if the condition (49) is satisfied and if there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ and a matrix $\overline{\mathbf{Z}} \in \mathbb{R}^{n\times(n+(r+1)m)}$ solution to the minimisation of γ under the following LMI constraint for $i = 1, \ldots, r$.

$$\begin{bmatrix} \mathbf{I}_n - \mathbf{P} & \mathbf{0} & \mathbf{\Phi}_{i,1}^T \\ \mathbf{0} & -\gamma^2\mathbf{I} & \mathbf{\Phi}_{i,2}^T \\ \mathbf{\Phi}_{i,1} & \mathbf{\Phi}_{i,2} & -\mathbf{P} \end{bmatrix} < 0 \quad (57)$$

where $\overline{\mathbf{\Phi}}_{i,1}$ and $\overline{\mathbf{\Phi}}_{i,2}$ are given by

$$\overline{\mathbf{\Phi}}_{i,1} = \mathbf{P}\overline{\mathbf{Y}\mathbf{X}}^{+}\mathbf{Y}_i + \overline{\mathbf{Z}\mathbf{X}}^{\perp}\mathbf{Y}_i$$

$$\overline{\mathbf{\Phi}}_{i,2} = \mathbf{P}\overline{\mathbf{Y}\mathbf{X}}^{+}\overline{\mathbf{Y}}_i + \overline{\mathbf{Z}\mathbf{X}}^{\perp}\overline{\mathbf{Y}}_i$$

Proof: The proof follows the lines of the proof of Corollary 2 and Theorem 3 and is therefore omitted. ■

Finally the design of the observer for continuous-time (resp. discrete-time) TSDS is reduced to the following procedure.

Step 1. Verify the existence condition (49).

Step 2. Solve the LMI (50) (resp. (57)) in $\mathbf{P}$ and $\overline{\mathbf{Z}}$.

Step 3. Compute $\mathbf{Z}$ with $\mathbf{Z} = \mathbf{P}^{-1}\overline{\mathbf{Z}}$. For a given $\mathbf{Z}$, $\mathbf{\Theta}$ is given by (54), then the matrices $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{K}_i$ are obtained. The matrices $\mathbf{N}_i$, $\mathbf{M}_i$ and $\mathbf{L}_i$ are derived from (19), (23) and (24) respectively.

VI. Application to fault diagnosis

In this section the UI decoupling and attenuating observers are used to perform fault diagnosis. Consider a continuous-time TSDS affected by faults $\boldsymbol{f}(t) \in \mathbb{R}^{q}$ and disturbances $\boldsymbol{w}(t) \in \mathbb{R}^{\overline{q}}$ defined by

$$\mathbf{E}\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{w}(t))(\mathbf{A}_i\boldsymbol{x}(t) + \mathbf{B}_i\boldsymbol{u}(t) + \mathbf{D}_{fi}\boldsymbol{f}(t) + \mathbf{D}_{wi}\boldsymbol{w}(t))$$

$$\boldsymbol{y}(t) = \mathbf{C}\boldsymbol{x}(t) + \mathbf{G}_f\boldsymbol{f}(t) + \mathbf{G}_w\boldsymbol{w}(t)$$

In the discrete-time case, the TSDS affected by faults $\boldsymbol{f}_k \in \mathbb{R}^{q}$ and disturbances $\boldsymbol{w}_k \in \mathbb{R}^{\overline{q}}$ is defined by

$$\mathbf{E}\boldsymbol{x}_{k+1} = \sum_{i=1}^{r} h_i(\boldsymbol{w}_k)\left(\mathbf{A}_i\boldsymbol{x}_k + \mathbf{B}_i\boldsymbol{u}_k + \mathbf{D}_{fi}\boldsymbol{f}_k + \mathbf{D}_{wi}\boldsymbol{w}_k\right)$$

$$\boldsymbol{y}_k = \mathbf{C}\boldsymbol{x}_k + \mathbf{G}_f\boldsymbol{f}_k + \mathbf{G}_w\boldsymbol{w}_k$$

It is assumed that each component of the disturbance vector is bounded, and that the value of this bound is known : $|w_i(t)| < \nu_i$ (resp. $|w_{ik}| < \nu_i$ in the discrete-time case) for $i = 1, \dots, \overline{q}$) for all t (resp. for all k). The well known generalised observer scheme [13] can be applied to propose a method for the fault diagnosis of TSDS. In this approach, q UIDAO are designed. The ℓ^{th} UIDAO is designed by considering the ℓ^{th} fault as an UI. A subset, denoted Σ_ℓ, of the disturbances can also be considered as UI provided the existence condition (49) is satisfied. In other words, the ℓ^{th} observer is designed for the system (47-48) (resp. (55-56)), with

$$\mathbf{D}_i = \left[\ \mathbf{D}_{fi}^{\boldsymbol{\ell}} \middle| \mathbf{D}_{wi}^{j},\ j \in \Sigma_\ell\ \right] \quad \overline{\mathbf{D}}_i = \left[\mathbf{D}_{wi}^{j},\ j \in \overline{\Sigma}_\ell\right]$$

$$\mathbf{G} = \left[\ \mathbf{G}_{f}^{\ell} \middle| \mathbf{G}_{w}^{j},\ j \in \Sigma_\ell\ \right] \quad \overline{\mathbf{G}} = \left[\mathbf{G}_{w}^{j},\ j \in \overline{\Sigma}_\ell\right]$$

where $\mathbf{M}^{\boldsymbol{\ell}}$ denotes the ℓ^{th} column of the matrix $\mathbf{M}$, and $\overline{\Sigma}_\ell$ denotes the complementary to Σ_ℓ in $\{1, 2, \dots, \overline{q}\}$.

As a consequence, the output estimation of the ℓ^{th} observer will be sensitive to all the faults but the ℓ^{th}, insensitive to the ℓ^{th} fault and a to subset of the disturbances, denoted Σ_ℓ, and maximally robust to the other disturbances belonging to the subset $\bar{\Sigma}_\ell$. The subsets of the UI are determined so that the decoupling condition (49) is satisfied for all the disturbances in Σ_ℓ and so that the $\mathcal{L}_2$-gain from the disturbances in $\bar{\Sigma}_\ell$ to the output estimation error is minimised. In other words, the output estimation error is a residual signal. A classical method for observer-based fault diagnosis is to suppose the occurrence of the ℓ^{th} fault if all residual signals, except the ℓ^{th}, are significantly different from zero. The problem is then to quantify the term *significantly*. In order to discriminate between the influence of the disturbances and the ℓ^{th} fault, one can compute the $\mathcal{L}_2$-gain from the disturbances to each output estimation error, as described in the following procedure.

For each fault $f_\ell(t)$ (resp. $f_{\ell k}$ in the discrete-time case)

• design the UIDAO, sensitive to all faults except $\boldsymbol{f}_\ell(t)$ (resp. $\boldsymbol{f}_{\ell k}$), insensitive to $\boldsymbol{w}_i(t)$ (resp. $\boldsymbol{w}_{ik}$) $i \in \Sigma_\ell$ and maximally robust to $\boldsymbol{w}_i(t)$ (resp. $\boldsymbol{w}_{ik}$), $i \in \overline{\Sigma}_\ell$.

• compute the norm-bound of the attenuated disturbances, denoted ρ_ℓ

$$\rho_\ell = \sqrt{\sum_{i \in \overline{\Sigma}_\ell} \nu_i^2}$$

• for each component of the output $\boldsymbol{y}_j(t)$ (resp. $\boldsymbol{y}_{jk}$), compute the $\mathcal{L}_2$-gain, denoted $g_{\ell j}$, from the attenuated disturbances $\{\boldsymbol{w}_i(t) \mid i \in \overline{\Sigma}_\ell\}$ (resp. $\{\boldsymbol{w}_{ik} \mid i \in \overline{\Sigma}_\ell\}$) to the j^{th} output estimation error and compute the boolean vector $\boldsymbol{b}_\ell(t) = \begin{bmatrix} b_{\ell 1}(t) & b_{\ell 2}(t) & \dots & b_{\ell m}(t) \end{bmatrix}$ (resp. $\boldsymbol{b}_{\ell k} = \begin{bmatrix} b_{\ell 1k} & b_{\ell 2k} & \dots & b_{\ell mk} \end{bmatrix}$), where $b_{\ell j}(t)$ (resp. $b_{\ell jk}$) is defined by

$$b_{\ell j}(t) = \begin{cases} 1, & \text{if } |\hat{\boldsymbol{y}}_{\ell j}(t) - \boldsymbol{y}_j(t)| > \alpha g_{\ell j} \rho_\ell \\ 0, & \text{else} \end{cases}$$

$$b_{\ell jk} = \begin{cases} 1, & \text{if } |\hat{\boldsymbol{y}}_{\ell jk} - \boldsymbol{y}_{jk}| > \alpha g_{\ell j} \rho_\ell \\ 0, & \text{else} \end{cases}$$

where $\hat{\boldsymbol{y}}_{\ell j}(t)$ (resp. $\hat{\boldsymbol{y}}_{\ell jk}$) is the j^{th} component of the estimated output given by the ℓ^{th} observer. The positive scalar α allows the designer to handle the compromise between non detection and false alarm (e.g. considering the accuracy of the model).

• compute the alarm $a_\ell(t)$ (resp. $a_{\ell k}$), affected to $\boldsymbol{f}_\ell(t)$ (resp. $\boldsymbol{f}_{\ell k}$), defined in the continuous-time case by

$$a_\ell(t) = \begin{cases} 1, & \text{if } (\boldsymbol{b}_i(t)\boldsymbol{b}_i^T(t) \geq 1,\ \forall i \neq \ell) \\ & \ \& \ (\boldsymbol{b}_\ell(t)\boldsymbol{b}_\ell^T(t) = 0) \\ 0, & \text{else} \end{cases}$$

or in the discrete-time case by

$$a_{\ell k} = \begin{cases} 1, & \text{if } (\boldsymbol{b}_{ik}\boldsymbol{b}_{ik}^T \geq 1,\ \forall i \neq \ell) \ \& \ (\boldsymbol{b}_{\ell k}\boldsymbol{b}_{\ell k}^T = 0) \\ 0, & \text{else} \end{cases}$$

This approach can be conservative, since the only available information about the disturbances is their amplitude bound and the $\mathcal{L}_2$-gain of their influence onto the output estimation error, thus it may imply non detection. This effect can be limited by the use of the parameter α which can be adjusted according to measurements of the system under healthy operation. Nevertheless, comparing each component of the estimation error with a threshold makes it possible to avoid distributing a significant error affecting a component on all the various components, and then reduces the non detection.

One should note that the generalised observer scheme is an efficient structure of diagnosis in order to detect and isolate single faults. In the case of simultaneous faults, two faults may cause non zero residue responses in all observers. In this case, the dedicated observer scheme can be considered as an alternative, but the decoupling conditions become much more restrictive since all the fault inputs but the ℓ^{th} have to be decoupled from the ℓ^{th} residue. This scheme is not detailed here, but can be readily applied since it suffices to change the definition of the matrices $\mathbf{D}_i$, $\overline{\mathbf{D}}_i$, $\mathbf{G}$ and $\overline{\mathbf{G}}$: the ℓ^{th} observer should be designed for the system (47-48) (resp. (55-56)), with

$$\mathbf{D}_i = \left[\mathbf{D}_{fi}^1 \ \dots \ \mathbf{D}_{fi}^{\ell-1} \ \mathbf{D}_{fi}^{\ell+1} \ \dots \ \mathbf{D}_{fi}^q \middle| \mathbf{D}_{wi}^j, \ j \in \Sigma_\ell \right]$$
$$\mathbf{G} = \left[\mathbf{G}_{fi}^1 \ \dots \ \mathbf{G}_{fi}^{\ell-1} \ \mathbf{G}_{fi}^{\ell+1} \ \dots \ \mathbf{G}_{fi}^q \middle| \mathbf{G}_{w}^j, \ j \in \Sigma_\ell \right]$$
$$\overline{\mathbf{D}}_i = \left[\mathbf{D}_{wi}^j, \ j \in \overline{\Sigma}_\ell\right] \quad , \quad \overline{\mathbf{G}} = \left[\mathbf{G}_{w}^j, \ j \in \overline{\Sigma}_\ell\right]$$

VII. Design example

In this section, the proposed approach for fault diagnosis is illustrated. Consider a discrete-time TSDS defined by

$$\mathbf{E}\boldsymbol{x}_{k+1} = \sum_{i=1}^{2} h_i(\boldsymbol{w}_k) \left(\mathbf{A}_i\boldsymbol{x}_k + \mathbf{B}_i\boldsymbol{u}_k + \mathbf{D}_{fi}\boldsymbol{f}_k + \mathbf{D}_{wi}\boldsymbol{w}_k\right)$$
$$\boldsymbol{y}_k = \mathbf{C}\boldsymbol{x}_k$$

with $\mathbf{E} = \mathrm{diag}\begin{pmatrix}1 & 1 & 1 & 0\end{pmatrix}$ and

$$\mathbf{A}_1 = \begin{pmatrix} -0.5 & -0.5 & 0.2 & 0.2 \\ -0.9 & 0.1 & 0.4 & 0.7 \\ -0.2 & -0.7 & 0 & 0 \\ -0.2 & -0.4 & 0.4 & 0 \end{pmatrix}, \ \mathbf{B}_1 = \mathbf{D}_{f1} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix},$$
$$\mathbf{A}_2 = \begin{pmatrix} 0.1 & -0.2 & 0.4 & 0.9 \\ -0.2 & 0.6 & -0.2 & -0.7 \\ 0.5 & -0.7 & -0.7 & 0.6 \\ -0.7 & 0.4 & 0.4 & 3.6 \end{pmatrix}, \ \mathbf{B}_2 = \mathbf{D}_{f2} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix},$$
$$\mathbf{D}_{w1} = \begin{pmatrix} 0 & 0 \\ 0.8 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \mathbf{D}_{w2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0.1 \\ 1 & 0 \end{pmatrix},$$
$$\mathbf{C} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

One can notice that the subsystem $(\mathbf{E}, \mathbf{A}_1)$ is impulsive. The finite spectrum of the two subsystems are significantly different since we have $\sigma_f(\mathbf{E}, \mathbf{A}_1) = \{-0.080, -0.564\}$ and $\sigma_f(\mathbf{E}, \mathbf{A}_2) = \{-0.935, 0.126, 0.996\}$, thus the global system is not close to a linear system. The sampling time is $t_s = 0.03$ s. The activating functions $h_i(\boldsymbol{w}_k)$ are defined by $h_{1k} = (1 + \tanh(u_{1k}/10))/2$ and $h_{2k} = 1 - h_{1k}$. The disturbances w_{1k} and w_{2k} are bounded centred white noise, with norm bound $\nu_1 = \nu_2 = 1$. The fault signals represent control input dysfunctions and they are defined by

$$f_{1k} = \begin{cases} -0.8\ u_{1k} & \text{if } 35 \leq t_k \leq 40 \\ 0 & \text{else} \end{cases} \tag{58}$$

$$f_{2k} = \begin{cases} -0.8\ u_{2k} & \text{if } 40 \leq t_k \leq 45 \\ 0 & \text{else} \end{cases} \tag{59}$$

The first UIADO is designed with the first control input as UI, the $\mathcal{L}_2$-gain from $\boldsymbol{w}$ to the first and second output estimation error are $g_{11} = 0.098$ and $g_{12} = 0.050$ respectively. The second UIADO is designed with the second control input and the second disturbance as UI, the $\mathcal{L}_2$-gain from w_1 to the first and second output estimation error are $g_{21} = 0.052$ and $g_{22} = 0.009$ respectively.

On figure 1, the inputs and the activating functions are displayed. Figures 2 and 3 display the comparison of the state variables and their estimates supplied by the UIDAO insensitive to the first fault. The fault f_{1k} appearing between $t_k = 35$ s and $t_k = 40$ s does not affect the estimation, whereas the estimation is sensitive to the fault f_{2k} present between $t_k = 40$ s and $t_k = 45$ s. Figures 4 and 5 display the comparison of the state variables and their estimates supplied by the UIDAO insensitive to the fault f_{2k} and affected by f_{1k}. The residual signal and their corresponding threshold, for $\alpha = 1$, are displayed on figures 6 and 7.

The residual signals computed with the output estimation error of the first observer are sensitive to f_{2k} and insensitive to f_{1k} whereas the residual signals computed with second UIADO are sensitive to f_1 and insensitive to f_2. The $\mathcal{L}_2$-gains g_{11}, g_{12}, g_{21} and g_{22} are good thresholds for fault isolation, with $\alpha = 1$, since the fault f_{2k} (occurring for $40 \leq t_k \leq 45$) is isolated at $t = 41.5$ s, and the fault f_{1k} (occurring for $35 \leq t_k \leq 40$) is isolated at $t = 35.1$ s. The sudden appearance or disappearance of a fault may cause abrupt changes of the state variables that the estimate cannot follow instantaneously. Thus even if the estimation is decoupled from the occurring fault, a residual signal may be transiently higher than the threshold. This phenomenon appears on figure 7 at $t = 45$ when f_{2k} disappears and causes a brief overshot of the output estimation error.

VIII. Conclusion

In this paper a simple method is proposed to design unknown input observers for Takagi-Sugeno descriptor systems. Sufficient existence conditions were given, and the determination of the observer parameters is based on solving a system of strict LMI. If the unknown input decoupling condition is not satisfied, it has been proposed to design an $\mathcal{L}_2$-observer in order to minimise the $\mathcal{L}_2$-gain from the UI to the estimated state. A compromise between perfect unknown input decoupling and unknown input attenuation can be made to design observer ensuring perfect decoupling face to a subset of the unknown inputs, and robustness face to the other unknown inputs. The three observer designs are treated in both continuous and discrete-time cases. The proposed observers are used to perform fault diagnosis. Designing a bank of observers, where each observer considers a fault as an unknown input, the generalised observer scheme can be extended to Takagi-Sugeno descriptor systems.

REFERENCES

[1] Akhenak, A., Chadli, M., Ragot, J., and Maquin, D.: ”State Estimation via Multiple Observer with Unknown Input. Application to the Three-Tank System”, in *Proc. of the* 5^{th} *IFAC SAFEPROCESS*, Washington, USA, 2003, pp. 245-251

[2] Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V.: *Linear Matrix Inequalities in System and Control Theory*, (SIAM studies in applies mathematics, 1994)

[3] Chen, J., and Patton, R.J.: *Robust Model-Based Fault Diagnosis for Dynamical Systems* (Kluwer Academic Pub., London, 2003)

[4] Dai, L.: *Singular Control Systems* (Springer, Berlin, 1989)

[5] Darouach, M., Zasadzinski, M., and Xu, S.J.: ”Full-order observers for linear systems with unknown inputs”, *IEEE Trans. Autom. Contr.*, 1994, 39, pp. 606-609

[6] Darouach, M., Zasadzinski, M., and Hayar, M.: ”Reduced-Order Observer Design for Descriptor Systems with Unknown Inputs”, *IEEE Trans. Autom. Contr.*, 1996, 41, pp. 1068-1072

[7] Darouach, M., and Boutayeb, M.: ”Design of observers for descriptor systems”, *IEEE Trans. Autom. Contr.*, 1995, 40, pp. 1323-1327

[8] Kawaji, S., and Kim, H.S.: ”Full Order Observer for Linear Descriptor Systems with Unknown-Inputs”, in *Proc. of the Conf. on Decision and Contr.*, New Orleans, USA, 1995, pp. 2366-2368

[9] Koenig, D.: ”Unknown Input Proportional Multiple-Integral Observer Design for Linear Descriptor Systems: Application to State and Fault Estimation”, *IEEE Trans. Autom. Contr.*, 2005, 50, pp. 212-217

[10] Koenig, D.: ”Observer Design for Unknown Input Nonlinear Descriptor Systems via Convex Optimization”, *IEEE Trans. Autom. Contr.*, 2006, 51, pp. 1047-1052

[11] Koenig, D., and Mammar, S.: ”Design of Proportional Integral Observer for Unknown Input Descriptor Systems,”, *IEEE Trans. Autom. Contr.*, 2002, 47, pp. 2057-2062

[12] Lian, K.Y., Liu, P., Liou, J.J., and Wu, T.C.: ”Robust Output Feedback Control for Fuzzy Descriptor Systems”, in *Proc of the IEEE Int. Fuzzy Syst. Conf.*, Melbourne, Australia, 2001, pp. 904-907

[13] Patton, R.J., Frank, P., and Clark, R.: *Fault Diagnosis in Dynamic Systems - Theory and Application* (Prentice Hall, London, 1989)

[14] Takagi, T., and Sugeno, M.: ”Fuzzy Identification of Systems and its Application to Modelling and Control”, *IEEE Trans. Syst., Man, Cybern.*, 1985, 15, pp. 116-132

[15] Tanaka, K., Ikeda, T., and Wang, H.O.: ”Fuzzy Regulators and Fuzzy Observers: Relaxed Stability Conditions and LMI-Based Designs”, *IEEE Trans. Fuzzy Syst.*, 1998, 6, pp. 250-265

[16] Tanaka, T., Hori, T., and Wang, H.O.: ”A multiple Lyapunov function approach to stabilization of fuzzy control systems”, *IEEE Trans. Fuzzy Syst.*, 2003, 11, pp. 582-589

[17] Taniguchi, T., Tanaka, K., Yamafuji, K., and Wang, H.O.: ”Fuzzy Descriptor Systems : Stability Analysis and Design via LMIs ”, in *Proc. of the Amer. Contr. Conf.*, San Diego, USA, 1999, 3, pp. 1827-1831

[18] Taniguchi, T., Tanaka, K., and Wang, H.O.: ”Fuzzy Descriptor Systems and Nonlinear Model Following Control”, *IEEE Trans. Fuzzy Syst.*, 2000, 8, pp. 442-452

[19] Wang, H.O., Tanaka, K., and Griffin, M.F.: ”An Approach to Fuzzy Control of Nonlinear Systems : Stability and Design Issues”, *IEEE Trans. Fuzzy Syst.*, 1996, 4, pp. 14-23

[20] Wang, Y., Zhang, Q.L., and Liu, W.Q.: ”Stability Analysis and Design for T-S Fuzzy Descriptor Systems”, in *Proc. of the IEEE Conf. Decision Contr.*, Orlando; USA, 2001, 4, pp. 3962-3967

[21] Yoneyama, J., and Ichikawa, I.: ”H_∞-Control for Takagi-Sugeno Fuzzy Descriptor Systems”, in *Proc of the IEEE Int. Conf. Syst. Man Cybern.*, Tokyo, Japan, 1999, 3, pp. 28-33

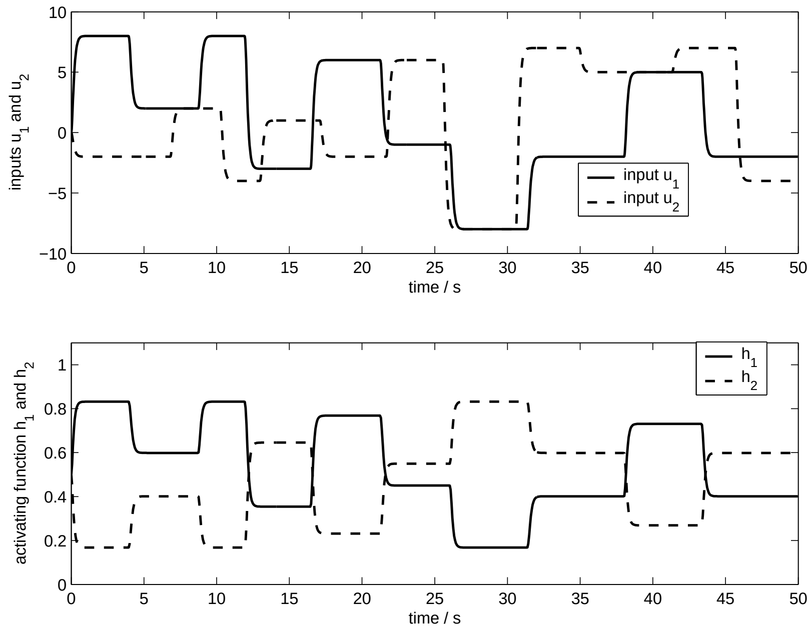

Fig. 1. Inputs and activating functions of the simulated system

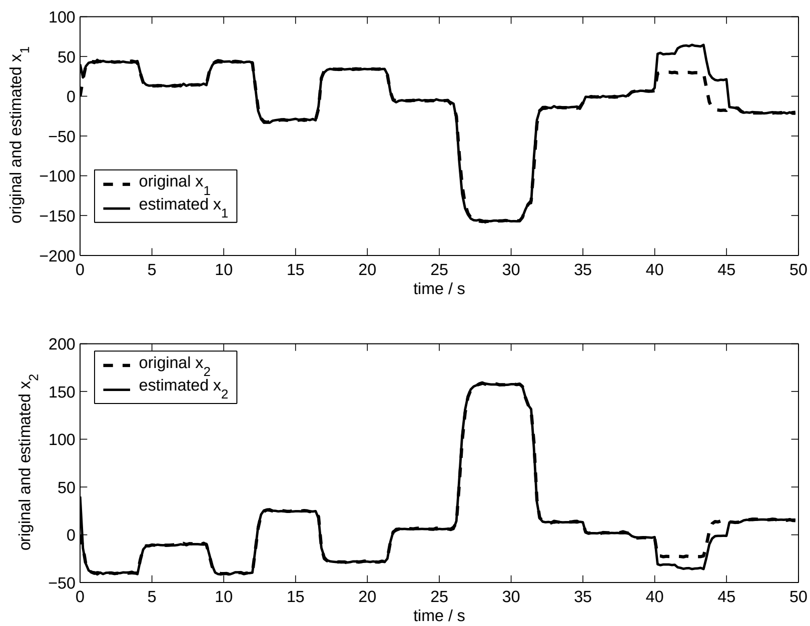

Fig. 2. Original and estimated x_{1k} and x_{2k} obtained with an observer decoupling the first input

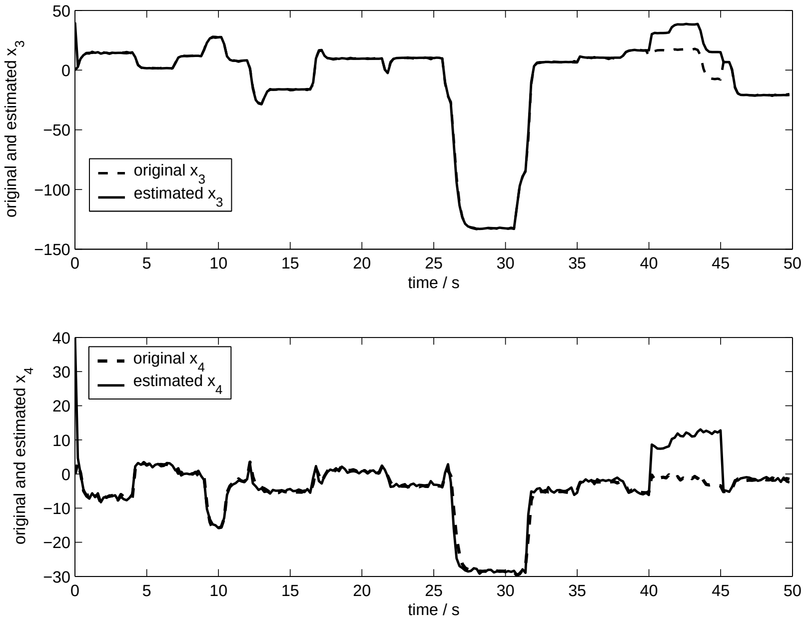

Fig. 3. Original and estimated x_{3k} and x_{4k} obtained with an observer decoupling the first input

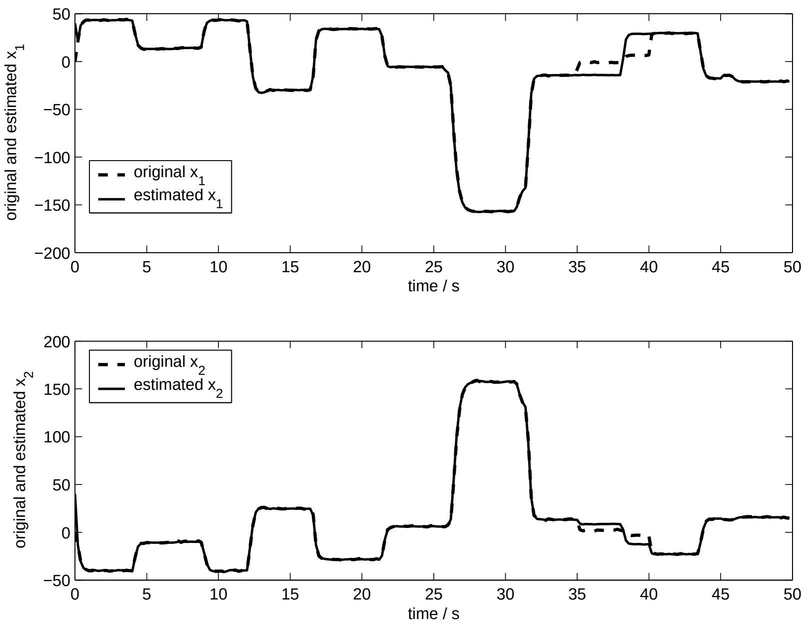

Fig. 4. Original and estimated x_{1k} and x_{2k} obtained with an observer decoupling the second input

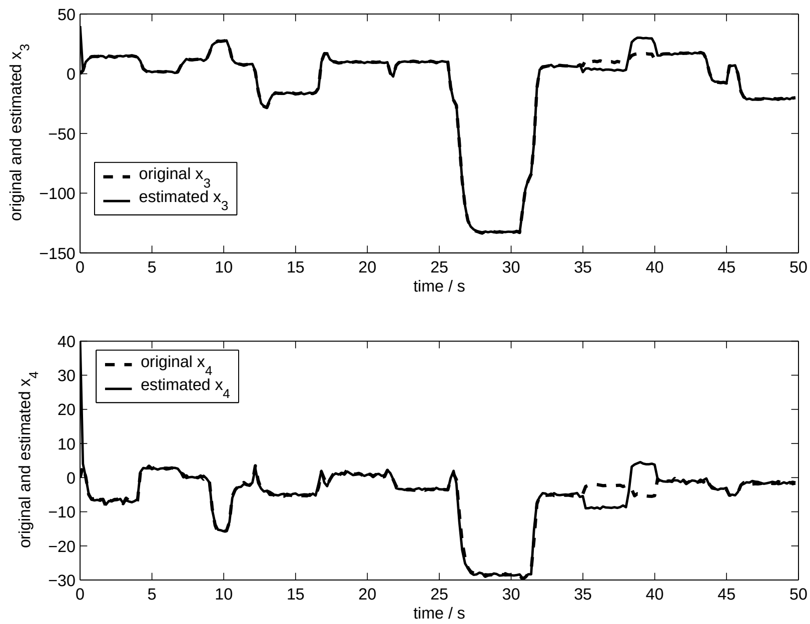

Fig. 5. Original and estimated x_{3k} and x_{4k} obtained with an observer decoupling the second input

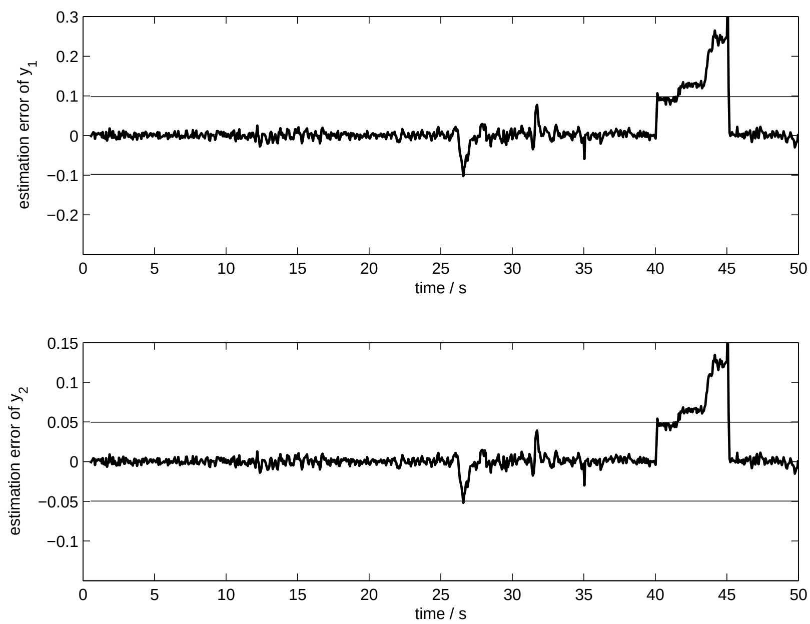

Fig. 6. Output estimation errors obtained with an observer decoupling the first input

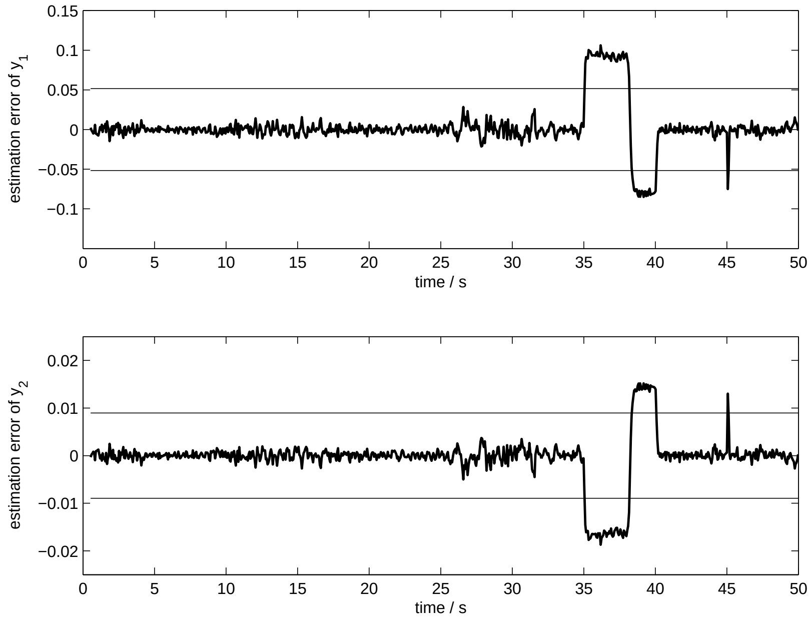

Fig. 7. Output estimation errors obtained with an observer decoupling the second input