OPTIMAL SENSOR/ACTUATOR LOCATION FOR DESCRIPTOR SYSTEMS USING LYAPUNOV-LIKE EQUATIONS

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Abstract: a method for optimal sensor and actuator location is proposed for the linear time-invariant descriptor systems. The actuators are selected to minimize the input energy needed to reach a given state and the sensor are selected to maximize the energy transmitted from the state variables to the outputs. Since controllability and observability can be quantified by the corresponding gramians, the problem of optimal actuator and sensor location is linked to the maximization of the gramians. Due to the finiteness of the possible locations for actuators and sensors this problem can be efficiently solved by using integer optimization tools. An illustrative example is provided.
Keywords: optimal sensor/actuator location, descriptor systems, gramians, Lyapunov-like equations

1. Introduction

Descriptor systems appear in many fields of system design and control such as power systems, electrical networks, hydraulic networks, even economic or biological modeling and thus motivate an important literature since [2]. As for non singular systems, one of the main issues is to efficiently command and observe the evolution of the systems, thus particular attention must be paid to actuator and sensor (A/S) location. Many techniques exist for non singular systems and are reviewed in [7], the aim of this paper is to generalize the one presented in [4], using gramians for singular systems studied in [1] and [8]. The paper is organized as follows, in the first section some useful results concerning descriptor systems are reminded, in the second section, gramians are defined and new Lyapunov-like equations are established. A methodology for A/S location is proposed in the third section and an illustrative example is given before concluding.

2. Preliminaries

Consider a linear time-invariant (LTI) descriptor system given by

$$\begin{cases} E\dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

where x, u and y are respectively the state variables, the inputs and the measured outputs; and E, A, B and C are real known constant matrices with $rank(E)<dim(x) = n$. Without loss of generality, E and A are assumed to be square matrices. Let introduce the Weierstrass-Kronecker decomposition of a pencil matrix [3]. Provided (E,A) is regular (*i.e.* $det(sE\text{-}A) \neq 0$), there exist two nonsingular matrices P and Q such that $PEQ=diag(I_{n1}, N)$ and $PAQ=diag(J, I_{n2})$ where N is nilpotent (its nilpotency index is denoted h), the eigenvalues of J are the finite eigenvalues of (E, A) and $n_1+n_2 = n$. The system (1) is equivalent to

$$\begin{cases} \dot{x}_1 = Jx_1 + B_1u \\ N\dot{x}_2 = x_2 + B_2u \\ y = C_1x_1 + C_2x_2 \end{cases} \tag{2}$$

with $PB = [B_1^T B_2^T]^T$, $CQ=[C_1\ \ C_2]$ and $x=Q[x_1^T\ x_2^T]^T$. The subsystems (I, J, B_1, C_1) and (N, I, B_2, C_2) are called causal and noncausal respectively, the corresponding subspaces are denoted H_c and H_{nc}.

The system (1) is stable if and only if its causal subsystem is stable. A system (1) is controllable (*resp.* observable) if and only if both causal and noncausal subsystems are controllable (*resp.* observable)

Any regular descriptor system can be uniquely defined by its series expansion using its Laurent parameters Φ_k [1]

$$(sE - A)^{-1} = \sum_{k\geq -h} \Phi_k s^{-k-1} \text{ with } \Phi_k = \begin{cases} Q\begin{pmatrix} J^k & 0 \\ 0 & 0 \end{pmatrix}P, & k \geq 0 \\ Q\begin{pmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{pmatrix}P, & k < 0 \end{cases}$$

which is valid in some set $0 < |s| \leq R$, for some $R > 0$. It is to notice that $\Phi_0 E$ and $-\Phi_{-1}A$ are the projection on H_c along H_{nc} and on H_{nc} along H_c respectively. Others properties of the Laurent parameters are given in [1] and [9].

Remark. In the impulse free case, $N=0$.

3. Gramians and Lyapunov-like equations

Definition [1] For LTI descriptor systems (E, A, B, C) the controllability gramian G_c is decomposed in a causal gramian (provided the integral exists) and a noncausal gramian denoted R_c and R_{nc} respectively and defined by

$$R_c = \int_0^\infty \Phi_0 e^{A\Phi_0 t} BB^T e^{\Phi_0^T A^T t} \Phi_0^T \, dt, \quad R_{nc} = \sum_{k=-h}^{-1} \Phi_k BB^T \Phi_k^T$$

$$G_c = R_c + R_{nc}$$

Analogous definitions stand for the observability gramian, G_o, decomposed in a causal and a non causal observability gramian, denoted O_c and O_{nc} respectively and defined by

$$O_c = \int_0^\infty \Phi_0^T e^{\Phi_0^T A^T t} C^T C e^{A\Phi_0 t} \Phi_0 \, dt, \quad O_{nc} = \sum_{k=-h}^{-1} \Phi_k^T C^T C \Phi_k$$

$$G_o = O_c + O_{nc}$$

Lyapunov-like equations for descriptor systems are given and established in the following theorem.

Theorem 1 (see proof in [6])

i. If R_c, R_{nc} and G_c exist they satisfy respectively

$$\Phi_0 A R_c + R_c A^T \Phi_0^T + \Phi_0 BB^T \Phi_0^T = 0 \tag{3}$$

$$\Phi_{-1} E R_{nc} E^T \Phi_{-1}^T - R_{nc} + \Phi_{-1} BB^T \Phi_{-1}^T = 0 \tag{4}$$

$$\begin{aligned} &(\Phi_0 + \Phi_{-1}/2)AG_c + G_c A^T (\Phi_0 + \Phi_{-1}/2)^T \\ &+ \Phi_{-1} E G_c E^T \Phi_{-1}^T + \Phi_0 BB^T \Phi_0^T + \Phi_{-1} BB^T \Phi_{-1}^T = 0 \end{aligned} \tag{5}$$

ii. If (1) is stable, R_c is the unique projection on H_c of the solutions of (3), R_{nc} and G_c are the unique solutions of (4) and (5) respectively.

iii. If (1) is stable, (1) is controllable if and only if G_c is the unique positive definite solution of (5).

Dual results concerning the observability are given by the theorem 2.

Theorem 2 (see proof in [6])

i. If O_c, O_{nc} and G_o exist they satisfy respectively

$$\Phi_0^T A^T O_c + O_c A \Phi_0 + \Phi_0^T C^T C \Phi_0 = 0 \quad (6)$$

$$\Phi_{-1}^T E^T O_{nc} E \Phi_{-1} - O_{nc} + \Phi_{-1}^T C^T C \Phi_{-1} = 0 \quad (7)$$

$$\begin{aligned}&(\Phi_0 + \Phi_{-1}/2)^T A^T G_o + G_o A(\Phi_0 + \Phi_{-1}/2) \\ &+ \Phi_{-1}^T E^T G_o E \Phi_{-1} + \Phi_0^T C^T C \Phi_0 + \Phi_{-1}^T C^T C \Phi_{-1} = 0\end{aligned} \quad (8)$$

ii. If (1) is stable, O_c is the unique projection on H_c of the solutions of (6), O_{nc} and G_o are the unique solutions of (7) and (8) respectively.

iv. If (1) is stable, (1) is observable if and only if G_o is the unique positive definite solution of (8).

In the discrete time case, the gramians are defined by (9) and are the solutions of Lyapunov-like equations given in the theorems 1 and 2 of [9].

$$\begin{aligned}&R_c = \sum_{k=0}^{\infty} \Phi_k B B^T \Phi_k^T,\ R_{nc} = \sum_{k=-h}^{-1} \Phi_k B B^T \Phi_k^T \\ &G_c = R_c + R_{nc} \\ &O_c = \sum_{k=0}^{\infty} \Phi_k^T C^T C \Phi_k,\ O_{nc} = \sum_{k=-h}^{-1} \Phi_k^T C^T C \Phi_k \\ &G_o = O_c + O_{nc}\end{aligned} \quad (9)$$

Remark. In the discrete time case, since discrete Lyapunov equations hold for *Rc* and *Rnc* (*resp.* O_c and O_{nc}) it is possible to establish a global one for G_c (*resp.* G_o).

4. Optimal sensor/actuator location

The methodology for A/S placement is similar to the one used in [4] for non-singular systems. According to theorem 1 (*resp.* theorem 2), the system (1) is controllable (*reps.* observable) if and only if (5) (*resp.* (8)) has a positive definite solution.

Lemma 1 (see proof in [6])

In the discrete time case, the minimal input energy needed to bring a discrete-time LTI singular system ($Ex_{k+1}=Ax_k+Bu_k$, $y_k=Cx_k$) from an initial state $x_0=0$ to a given state X is given by (10), and the output energy of an input-free system with a given initial state X is given by (11)

$$\begin{aligned}&\min_u \frac{1}{2} \sum_{k \geq 0} u_k^T u_k = \frac{1}{2} X^T G_c^{-1} X, \\ &\text{with } \ x_0 = 0,\ x_k \underset{k \to \infty}{\to} X\end{aligned} \quad (10)$$

$$\sum_{k \geq 0} y_k^T y_k = X^T (QP)^{-T} G_o (QP)^{-1} X \quad (11)$$

Thus the problem of optimal A/S location can be considered as equivalent to the maximization of the gramians. The actuators and the sensors (thus the matrices *B* and *C*) must be set such that (5) and (8) have positive definite solution with highest possible trace.

Optimal placement of n_a actuators and n_s sensors is equivalent to maximize $Trace(G_c)$ and $Trace(G_o)$ with *B* and *C* verifying (12), thus is an integer programming problem.

$$\begin{aligned}&\sum_{j=1,\dots,n_a} b_{ij} = 1,\ \sum_{i=1,\dots,n} b_{ij} = 1,\ b_{ij} = 0 \text{ or } 1 \\ &\sum_{i=1,\dots,n_s} c_{ij} = 1,\ \sum_{j=1,\dots,n} c_{ij} = 1,\ c_{ij} = 0 \text{ or } 1\end{aligned} \quad (12)$$

For large scale systems, integer optimization tools (*e.g.* Branch and Bounds method, see [5]) may be used to reduced the computation complexity.

5. Illustrative example

Since the optimal location problems for sensors and actuators are dual, we only address the former.

Consider the LTI singular system described by

$$A = \begin{bmatrix} -11 & 1 & 400 & 0 & 0 \\ 3 & -249 & 300 & 3 & 0 \\ 0 & 300 & -602 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},\ E = \begin{bmatrix} 1&0&1&0&0 \\ 0&1&0&0&1 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The dimension of the non causal subspace is $n_2=2$, in order to ensure it is observable, the 4th state variable must be measured. A second sensor is necessary and sufficient to ensure the observability of the system, it can be placed on one of the three first states. Let compute the observability gramian in each case.

When x_1 and x_4 are measured, $Tr(G_o)=2773$, when x_2 and x_4 are measured $Tr(G_o)=1147$ and when x_3 and x_4 are measured $Tr(G_o)=842$.

Obviously the optimal solution is to place the sensors in order to measure x_1 and x_4, thus we have determined the matrix *C*

6. Conclusion

In this paper, using new Lyapunov-like equations, a method for optimal actuator (*resp.* sensor) location for descriptor systems is proposed and illustrated. It is shown that the problem of optimal location is equivalent to an integer programming problem, where the objective is to determine the matrix *B* (*resp.* *C*) that maximizes the energy transmitted from the input to the state (*resp.* from the state to the outputs) or equivalently to maximize the controllability (*resp.* observability) gramian.

7. References

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