Input-to-State Safety for Discrete-Time Nonlinear Systems: Sufficiency and Converse Results

AQ:1 Mayank Shekhar Jha and Bayu Jayawardhana, *Senior Member, IEEE* AQ:2

***Abstract*—Input-to-state safety (ISSf) extends comparison-function-based robustness analysis from stability to safety by quantifying degradation of safety margin under exogenous inputs. For continuous-time systems, sufficiency conditions for the existence of ISSf comparison function have been established, while its converse result remains an open problem. Also, the extension of the ISSf notion and its characterization to the discrete-time (DT) setting are largely unavailable. This letter develops ISSf notions for general discrete-time nonlinear systems. We introduce a global practical ISSf notion and two tube-local notions namely local ISSf and strong local ISSf properties. We show that an ISSf barrier certificate on a compact tube implies local ISSf property under an explicit state-input compatibility condition. Conversely, strong local ISSf property under bounded inputs implies the existence of an ISSf barrier certificate.**

***Index Terms*—Input to state safety, safe control design, Barrier functions, control barrier functions, converse results, safe control.**

I. Introduction

As AUTONOMOUS systems approach Level 5 (full autonomy), ensuring safe behavior across all subsystems has become increasingly imperative. This trend has led to the growing body of work in the area of safe control design and learning [1], [4]. From a control-theoretic perspective, safety is formally defined as the forward invariance property of a prescribed safe set [1], [9], [13]. Consequently, barrier certificates have become a standard tool for safety verification [9], typically providing sufficient conditions for set invariance. The integration of barrier certificates into control design, first introduced via Control Barrier Function (CBF) in [13], has since led to significant advancements in the literature on safe control analysis and synthesis. There are two common paradigms of safe control synthesis in the literature. The first is optimization-based synthesis where the barrier certificates are incorporated as inequality constraints. The seminal works of [1] integrate CBFs and Control Lyapunov Functions (CLFs) into a Quadratic Programming (QP) formulation that allows real-time safe control synthesis for many applications by balancing safety and performance in each optimization step. The second framework focuses on the unification of CLFs and CBFs leading to the development of control Lyapunov-Barrier functions (CLBFs) as explored in [11]. Beyond traditional control, these CBF-CLF-QP approaches have also been investigated in safe control learning. To name a few, the letter [6] presents CBF-CLF-QP for safe initialization and exploration, and the letter [3] uses CBFs to guarantee safe learning of optimal control law. In the aforementioned works, the safety certification of the nominal closed-loop systems does not immediately guarantee the systems' safety when perturbed by exogenous signals.

Analogous to the notion of input-to-state stability (ISS) [12], which has been very useful to quantify the robustness of systems' stability in the presence of external disturbance inputs, the notion of input-to-state safety (ISSf) as presented in [10] provides robustness quantification of systems' safety under exogenous inputs. The comparison-function characterization in ISS [12] is adapted to the systems' safety context in the ISSf notion for continuous-time (CT) systems. The ISSf inequalities are defined based on the safety margin $|x|_{\mathcal{D}}$. Instead of upper-bounding the state norm as in ISS, in ISSf inequalities, $|x|_{\mathcal{D}}$ is lower-bounded by a comparison term that depends on the initial safety margin and time, and it can be negatively influenced by the magnitude of exogenous input signal by eroding the certified safety margin. Correspondingly, the characterization of ISSf property using ISSf barrier certificates is further studied in [10]. While sufficient conditions on ISSf inequalities using ISSf barrier certificates are presented in these letters, the converse result remains an open problem. Moreover, the adaptation of this ISSf notion to the discrete-time (DT) systems is largely missing in the literature. A DT counterpart of the sufficiency result is non-trivial because the state transitions in DT systems can exceed the safety boundaries in one time step and the infinitesimal calculus

AQ:3 Received 16 March 2026; revised 23 May 2026; accepted 8 June 2026. This work was supported by Netherlands' Growth Fund Program AQ:4 of NXTGEN Hightech Equipments on Autonomous Factory. This article has not been presented at a conference. Recommended by Senior Editor A. P. Aguiar. *(Corresponding author: Mayank Shekhar Jha.)*

Mayank Shekhar Jha is with CRAN, CNRS, Université de Lorraine, AQ:5 France (e-mail: mayank-shekhar.jha@univ-lorraine.fr).

AQ:6 Bayu Jayawardhana is with the Engineering and Technology Institute Groningen, Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands (e-mail: b.jayawardhana@rug.nl).

Digital Object Identifier 10.1109/LCSYS.2026.3708562

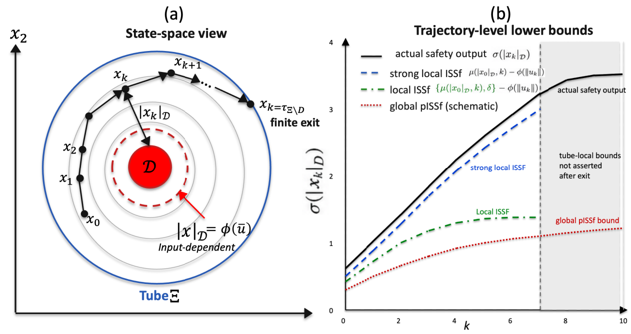

Fig. 1. (a) state-space view showing the unsafe set $\mathcal{D}$, a compact tube Ξ, the input-dependent clearance layer $|x|_{\mathcal{D}} = \phi(\bar{u})$ for $\|u_j\| \leq \bar{u}$ and a discrete-time trajectory that starts outside this layer, moves away from $\mathcal{D}$, and exits $\Xi \setminus \mathcal{D}$ in finite time. (b) schematic comparison-function view of the safety output $\sigma(|x_k|_{\mathcal{D}})$ and the certified lower bounds. The strong local ISSf bound grows until the tube-exit time $\tau_{\Xi\setminus\mathcal{D}}$, while the local ISSf bound is its saturated counterpart through δ. The global pISSf bound is a trajectory-level estimate involving the past-input norm $\|u\|_{\infty,[0,k)}$. The tube-local estimates are asserted only for $k < \tau_{\Xi\setminus\mathcal{D}}$; the shaded region indicates that they are no longer claimed after exit.

used for deriving ISSf property in CT systems is not directly applicable to the DT ones.

To address these two scientific gaps, this letter presents a variety of DT ISSf notions (see Fig. 1), where sufficient conditions of these notions using DT ISSf barrier certificates are presented, and subsequently, proves the converse result for the strongly local ISSf notion. Firstly, we introduce a global practical ISSf estimate for DT systems, which gives a trajectory-level robustness bound in terms of the truncated input norm. Secondly, we introduce local and strongly local ISSf notions using local tubes that encapsulate the unsafe set. This allows a natural construction of converse barrier certificates, which are valid locally around the unsafe set. Thirdly, we show that an ISSf barrier certificate yields a local ISSf estimate under a compatibility condition. Finally, we prove the converse result, namely, a strong local ISSf implies the existence of a practical ISSf barrier certificate.

In parallel, another set of literature studies related ISSf notion in the sense of [1], which emphasizes robust CBF formulations and use the term ISSf to describe disturbance-dependent enlargement of the safe set and QP-based filters. Readers are referred to [1], [4], and [8] for further reading about this approach that remains conceptually distinct in that it is primarily a robust forward-invariance property/synthesis condition respecting CBF constraints and QP filters, whereas the approach adopted here is akin to the ISS comparison inequality for systems' trajectories in the presence of exogenous input with some inherent differences as described above. Throughout this letter, we study the ISSf in the sense of [10].

Notations. Let $\mathbb{Z}_{\geq 0} := \mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0} := [0, \infty)$. For $S \subset \mathbb{R}^n$, $\mathrm{int}(S)$ and ∂S denote its interior and boundary. For a nonempty closed S, define $|x|_S := \inf_{\xi \in S} \|x - \xi\|$. For an input sequence $u = \{u_k\}_{k \in \mathbb{Z}_{\geq 0}}$, set $\|u\|_{\infty,[0,k)} := \max_{0 \leq i < k} \|u_i\|$, $\|u\|_{\infty,[0,0)} := 0$ and $\|u\|_\infty := \sup_{k \in \mathbb{Z}_{\geq 0}} \|u_k\|$. For $v \in \mathbb{U}$, write $x^+ = F(x, v)$ and define $\Delta W(x, v) := W(F(x, v)) - W(x)$. Comparison function classes $\mathcal{K}, \mathcal{K}_\infty, \mathcal{KL}$ are standard. A function $\mu : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KK}$ if $\mu(\cdot, k) \in \mathcal{K}$ for all $k \geq 1$, $\mu(\cdot, 0)$ is continuous with $\mu(0, 0) = 0$, and $\mu(s, \cdot)$ is non-decreasing for each fixed s.

II. Preliminaries and Problem Statement

We consider the DT nonlinear system for $k \in \mathbb{Z}_{\geq 0}$

$$x_{k+1} = F(x_k, u_k) \tag{1}$$

where $x_k \in \mathcal{X} \subset \mathbb{R}^n$ is the state, $u_k \in \mathbb{U} \subset \mathbb{R}^m$ is the exogenous input, and $F : \mathcal{X} \times \mathbb{U} \to \mathcal{X}$ is a continuous map. We assume input signal u is taken from the space of bounded signals $\mathcal{U}$ defined by

$$\mathcal{U} := \Big\{ u = \{u_k\}_{k \in \mathbb{Z}_{\geq 0}} : \ u_k \in \mathbb{U} \ \forall k, \ \|u\|_\infty < \infty \Big\}. \tag{2}$$

Let $\mathcal{D} \subset \mathcal{X}$ be a compact, unsafe set and let $\mathcal{X}_0 \subset \mathcal{X} \setminus \mathcal{D}$ denote a set of initial conditions. Given $\mathcal{X}_0$, a pair (x_0, u) is called *feasible* if $x_0 \in \mathcal{X}_0$ and $u \in \mathcal{U}$. We denote the semiflow of (1) for a given feasible initial condition and input (x_0, u) by $\pi_k(x_0, u)$ for all $k \geq 0$. Whenever it is clear from the context, we express $x_k = \pi_k(x_0, u)$ in the rest of the letter.

Barrier certificates can be used to certify forward invariance of sub-level sets [9], [13]. Given $B : \mathcal{X} \to \mathbb{R}$, let $\mathcal{S}_B := \{\xi : B(\xi) \leq 0\}$. For the DT system (1), all trajectories starting from $\mathcal{X}_0$ are certified safe for all $u \in \mathcal{U}$ if $\mathcal{X}_0 \subset \mathcal{S}_B$, $B > 0$ on $\mathcal{D}$, and $B\big(F(\xi, v)\big) \leq B(\xi), \forall \xi \in \mathcal{S}_B, \ \forall v \in \mathbb{U}$. We note that this robust one-step condition is conservative in DT because it requires invariance against every admissible input value, even near the unsafe boundary. In the following section, we introduce ISSf for DT systems that relaxes this all-or-nothing requirement by quantifying how the admissible input magnitude trades off with the current safety margin $|x_k|_{\mathcal{D}}$.

III. Input-to-State Safety for DT Systems

We introduce three ISSf notions: (i) global pISSf notion that gives a trajectory-level safety in terms of the truncated input norm; (ii) strong local ISSf notion that presents strengthened tube-local condition formalizing finite-exit time condition which is later used for the converse construction; and (iii) local ISSf notion, which is a saturated tube-local estimate recovered from ISSf barrier certificates.

A. Global Practical ISSf

Definition 1 (Global Practical ISSf): The DT system as in (1) is *globally practically input-to-state safe (pISSf) w.r.t. $\mathcal{D}$* if there exist $\sigma \in \mathcal{K}$, $\alpha, \phi \in \mathcal{KK}$, and a non-decreasing sequence $\{\gamma_k\}_{k \in \mathbb{Z}_{\geq 0}}$ such that for every $x_0 \in \mathcal{X} \setminus \mathcal{D}$, every $u \in \mathcal{U}$, and all $k \in \mathbb{Z}_{\geq 0}$,

$$\sigma(|x_k|_{\mathcal{D}}) \geq \alpha(|x_0|_{\mathcal{D}}, k) - \phi(\|u\|_{\infty,[0,k)}, k) - \gamma_k. \tag{3}$$

Then, as per this notion, the global ISSf is a special case of global pISSf by setting $\gamma_k \equiv 0$. The global pISSf notion is suited for trajectory-level safety assessment and robustness bounds, considering the cumulative worst-case effect of the inputs that have acted up to time k, namely $\|u\|_{\infty,[0,k)}$. It does not, however, consider the one-step conditions needed for barrier-certificate characterizations.

B. Local ISSf Properties

Let us consider $\Xi \subset \mathcal{X}$ as a closed and compact tube around $\mathcal{D}$. We define

$$r_\Xi := \sup_{x\in\Xi} |x|_{\mathcal{D}} \; < \; \infty. \tag{4}$$

and the corresponding boundary separation $r_{\partial\Xi} := \inf_{\xi\in\partial\Xi} |\xi|_{\mathcal{D}}$. With Ξ being compact and $\mathcal{D} \subset \mathrm{int}(\Xi)$, $r_{\partial\Xi} > 0$ holds and for the distance tube $\Xi = \{x \in \mathcal{X} : \; |x|_{\mathcal{D}} \le r_\Xi\}$ one has $r_{\partial\Xi} = r_\Xi$. To develop local properties of ISSf where converse results will be established, we restrict (1) to bounded inputs as $u \in \mathcal{U}_M$ by defining, for a constant $M > 0$,

$$U_M := \{v \in \mathbb{U} : \; \|v\| \le M\} \tag{5}$$

and the corresponding class of bounded input sequences

$$\mathcal{U}_M := \{u \in \mathcal{U} : \; u_k \in U_M \; \forall k \in \mathbb{Z}_{\ge 0}\}. \tag{6}$$

All tube-local results below assume a fixed bound M (i.e., $u \in \mathcal{U}_M$), which is needed for establishing uniform exit-time and value-function bounds. We propose a novel local tube level ISSf notion that captures the progression of safety (growth of distance from unsafe set) under bounded inputs.

Definition 2 (Strong Local ISSf): Let $\Xi \subset \mathcal{X}$ be a compact tube containing $\mathcal{D}$ and $M > 0$. The DT system (1) is *strongly ISSf locally in* Ξ *w.r.t.* $\mathcal{D}$ *for inputs in* $\mathcal{U}_M$ if there exist $\sigma, \phi \in \mathcal{K}$ and $\mu \in \mathcal{KK}$ such that, for every $s \in (0, r_\Xi]$

$$\bar{T}_\mu(s) := \min\Big\{k \in \mathbb{Z}_{\ge 1} : \; \mu(s,k) > \sigma(r_\Xi) + \phi(M)\Big\} < \infty, \tag{7}$$

and for every $x_0 \in \Xi \setminus \mathcal{D}$ and every $u \in \mathcal{U}_M$, the corresponding trajectory satisfies

$$\sigma\big(|x_k|_{\mathcal{D}}\big) \; \ge \; \mu\big(|x_0|_{\mathcal{D}}, k\big) \; - \; \phi\big(\|u_{k-1}\|\big) \tag{8}$$

for all $k \in \mathbb{Z}_{\ge 1}$ such that $x_j \in \Xi \setminus \mathcal{D}$ for all $j \in \{0, 1, \ldots, k\}$. Definition 2 strengthens local ISSf in that condition (7) does not require the certified lower bound $\mu(s,k)$ to diverge to infinity. It only requires it to exceed the finite threshold $\sigma(r_\Xi) + \phi(M)$ enforcing finite escape from the near-unsafe tube $\Xi \setminus \mathcal{D}$ under bounded inputs, allowing the system to plausibly enter a larger safe region and stabilize outside the tube.

Following [10], a pair $(x_0, u) \in (\Xi \setminus \mathcal{D}) \times \mathcal{U}_M$ is called *ISSf-admissible* for (8) if the RHS of (8) is strictly positive for all $k \in \mathbb{Z}_{\ge 1}$ such that $x_j \in \Xi \setminus \mathcal{D}$ for all $j \le k$.

The inequality (8) certifies growth of the safety margin while the trajectory remains in $\Xi \setminus \mathcal{D}$. Condition (7) only requires this certified lower bound to exceed the finite tube threshold $\sigma(r_\Xi) + \phi(M)$. Particularly, it does not imply unbounded growth of the trajectory outside the tube. Let us present a (weaker) local ISSf notion that considers a point-wise input gain $\phi(\|u_{k-1}\|)$ that further serves as a natural interface for tube-based analysis, which is the DT counterpart of that in [10].

Definition 3 (Local ISSf): Given a bounded tube $\Xi \subset \mathcal{X}$ of $\mathcal{D}$ in the sense of (4), the DT system (1) is *ISSf locally in* Ξ *w.r.t.* $\mathcal{D}$ *for inputs in* $\mathcal{U}_M$ if there exist $\sigma, \phi \in \mathcal{K}$, $\mu \in \mathcal{KK}$, and $\delta > 0$ such that for every $x_0 \in \Xi \setminus \mathcal{D}$ and every $u \in \mathcal{U}_M$, the trajectory x_k satisfies

$$\sigma\big(|x_k|_{\mathcal{D}}\big) \; \ge \; \min\{\mu(|x_0|_{\mathcal{D}}, k), \delta\} \; - \; \phi\big(\|u_{k-1}\|\big) \tag{9}$$

for all $k \in \mathbb{Z}_{\ge 1}$ such that $x_j \in \Xi \setminus \mathcal{D}$ for all $j \in \{0, 1, \ldots, k\}$. Inequality (9) is a tube-local ISS-type estimate for the safety output $|x|_{\mathcal{D}}$: before exit from $\Xi \setminus \mathcal{D}$, the certified margin at x_k is lower-bounded by a saturated transient term minus the preceding-step input penalty. Since Ξ is bounded, $\sigma(|x_k|_{\mathcal{D}})$ cannot grow unbounded while $x_k \in \Xi$. Therefore, strong local ISSf imposes a finite-exit time from $\Xi \setminus \mathcal{D}$ under bounded inputs as formalized in the following lemma.

Lemma 1 (Strong Local ISSf Implies Uniform Finite Exit): Assume system (1) is strongly ISSf locally in Ξ w.r.t. $\mathcal{D}$ for inputs in $\mathcal{U}_M$ in the sense of Definition 2. Then, for every $x_0 \in \Xi \setminus \mathcal{D}$ and every $u \in \mathcal{U}_M$, the exit time $\tau_{\Xi\setminus\mathcal{D}}(x_0, u) := \inf\{k \ge 0 : \; x_k \notin \Xi \setminus \mathcal{D}\}$ satisfies

$$\tau_{\Xi\setminus\mathcal{D}}(x_0, u) \le \bar{T}_\mu(|x_0|_{\mathcal{D}}),$$

where $\bar{T}_\mu$ is defined in (7). Moreover, $\bar{T}_\mu$ is non-increasing and locally bounded on $(0, r_\Xi]$.

Proof. Fix $x_0 \in \Xi \setminus \mathcal{D}$ and $u \in \mathcal{U}_M$, and set $s := |x_0|_{\mathcal{D}}$. Since $x_0 \in \Xi \setminus \mathcal{D}$, the exit time satisfies $\tau_{\Xi\setminus\mathcal{D}}(x_0, u) \ge 1$. For every integer k with $1 \le k < \tau_{\Xi\setminus\mathcal{D}}(x_0, u)$, one has $x_k \in \Xi$. Hence $\sigma(|x_k|_{\mathcal{D}}) \le \sigma(r_\Xi)$ and $\phi(\|u_{k-1}\|) \le \phi(M)$. It follows from (8) that $\mu(s,k) \le \sigma(r_\Xi) + \phi(M)$, for all $1 \le k < \tau_{\Xi\setminus\mathcal{D}}(x_0, u)$. By the definition of $\bar{T}_\mu(s)$, this is impossible for $k = \bar{T}_\mu(s)$, thus $\tau_{\Xi\setminus\mathcal{D}}(x_0, u) \le \bar{T}_\mu(s)$. Since $\mu(\cdot, k) \in \mathcal{K}$ for each $k \ge 1$, $\bar{T}_\mu(\cdot)$ is non-increasing, and hence locally bounded on $(0, r_\Xi]$. □

We note that strong local ISSf implies local ISSf since $\min\{\mu(s,k), \delta\} \le \mu(s,k)$ with the same previous-step input penalty $\phi(\|u_{k-1}\|)$. However, the converse is false in general as local ISSf can be saturated and does not imply finite exit-time from $\Xi \setminus \mathcal{D}$. The strong local notion is therefore used only for the converse construction later in Section VI.

Example 1: For a given $r \in (0,1)$, let us consider the following DT system $x_{k+1} = \lambda(x_k)x_k + u_k$, with $|u_k| \le M$ where $\lambda(s) = 1 - r$ for all $s \le 1 - r$, $\lambda(s) = 1 + r$ for all $s > 1 + r$ and $\lambda(s) = s$ elsewhere. In this case, $\mathcal{X} = \mathbb{R}$, $\mathcal{D} = [1 - \epsilon, 1 + \epsilon]$ with $\epsilon < r$. When $u_k \equiv 0$ one can immediately analyze that for all $x_0 < 1 - \epsilon$, $x_k \to 0$ as $k \to \infty$ with decay rate of at least $1 - \epsilon$, and for all $x_0 > 1 + \epsilon$, $x_k \to \infty$ as $k \to \infty$ with growth rate of at least $1 + \epsilon$. Note that the singleton $\{1\} \in \mathcal{D}$ is also an equilibrium point. We will show that this system is strongly ISSf locally in Ξ w.r.t. $\mathcal{D}$, where $\Xi = [1 - \xi, 1 + \xi]$ with $\xi \in (\epsilon, r)$. Note that $\Xi \setminus \mathcal{D} = [1 - \xi, 1 - \epsilon) \cup (1 + \epsilon, 1 + \xi]$. When $x_0 \in [1 - \xi, 1 - \epsilon)$, $x_{k+1} = \lambda(x_k)x_k + u_k \le (1 - \epsilon)x_k + |u_k|$. By comparison lemma, we have that $x_k \le (1 - \epsilon)^k x_0 + \sum_{i=0}^{k-1} (1 - \epsilon)^{k-1-i}|u_i|$. Since $|x_k|_{\mathcal{D}} = 1 - \epsilon - x_k$ on $[1 - \xi, 1 - \epsilon)$, it follows then that for all $k \ge 2$, $|x_k|_{\mathcal{D}} \ge 1 - \epsilon - (1 - \epsilon)^k x_0 - \sum_{i=0}^{k-1} (1 - \epsilon)^{k-1-i}|u_i| = |x_0|_{\mathcal{D}} + x_0(1 - (1 - \epsilon)^k) - M\left(\frac{1-(1-\epsilon)^{k-2}}{\epsilon}\right) - |u_{k-1}|$. Note that when $k = 1$, the right-hand side is simply given by $|x_0|_{\mathcal{D}} + x_0\epsilon - |u_0|$. If M is sufficiently small, e.g. $M < \epsilon^2(1 - \xi)$, we have the strongly ISSf inequality with $\mu(|x_0|_{\mathcal{D}}, k) = |x_0|_{\mathcal{D}} + x_0(1 - (1 - \epsilon)^k) - \epsilon(1 - \xi)$ and $\phi(|u_{k-1}|) = |u_{k-1}|$. Similar computation can be performed for the case of $x_0 \in (1 + \epsilon, 1 + \xi]$. In this case, $x_{k+1} = \lambda(x_k)x_k + u_k \ge (1 + \epsilon)x_k + u_k$. When $M < \epsilon$ so that $|u_k| \le M$ for all k, then $x_{k+1} \ge (1 + \epsilon - M)x_k$ holds with $(1 + \epsilon - M) > 1$. By comparison lemma, we have that $x_k \ge (1 + \epsilon - M)^k x_0$. Since $|x_k|_{\mathcal{D}} = x_k - (1 + \epsilon)$ on $(1 + \epsilon, 1 + \xi]$, it follows then that $|x_k|_{\mathcal{D}} \geq (1+\epsilon-M)^k x_0 - (1+\epsilon) = |x_0|_{\mathcal{D}} + ((1+\epsilon-M)^k - 1)x_0 \geq |x_0|_{\mathcal{D}} + ((1+\epsilon-M)^k-1)x_0 - |u_{k-1}|$. Hence, for the case $x_0 \in (1+\epsilon, 1+\xi]$, ISSf inequality holds with $\mu(|x_0|_{\mathcal{D},k}) = |x_0|_{\mathcal{D}} + ((1+\epsilon-M)^k-1)x_0$ and $\phi(|u_{k-1}|) = |u_{k-1}|$. Based on these computations, one can derive the uniform finite exit-time $\bar{T}_\mu(s)$ in Lemma 1. We note also that when $\mathcal{D}$ is restricted to a singleton with $\mathcal{D} = \{1\}$, rather than an interval $[1-\epsilon, 1+\epsilon]$ as before, we do not have a uniform finite exit time. Thus, in this case, we can have local ISSf property, but not the strong one as above. □

IV. ISSf Barrier Certificate

In this section we introduce an ISSf barrier certificate for DT systems on the tube Ξ, namely a certificate B that is radially comparable to $|x|_{\mathcal{D}}$ (via $\mathcal{K}_\infty$ bounds) and satisfies a strict ISS-type one-step dissipation inequality with input gain.

Definition 4 (ISSf Barrier Certificate on a Tube): A continuous function $B : \Xi \to \mathbb{R}$ is an *ISSf barrier certificate on* Ξ *for inputs in* U_M if there exist $\underline{\alpha}, \overline{\alpha}, \psi \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ such that for all $x \in \Xi \backslash \mathcal{D}$ and all $v \in U_M$ with $x^+ := F(x, v) \in \Xi \backslash \mathcal{D}$,

$$-\overline{\alpha}(|x|_{\mathcal{D}}) \;\leq\; B(x) \;\leq\; -\underline{\alpha}(|x|_{\mathcal{D}}), \qquad (10)$$

$$B(x^+) - B(x) \;\leq\; -\psi(|x|_{\mathcal{D}}) \;+\; \rho(\|v\|). \qquad (11)$$

Note that this ISSf barrier uses a Lyapunov-like sign convention, where $B < 0$ on $\Xi \setminus \mathcal{D}$ and approaches 0 at $\mathcal{D}$, in contrast to the separating barrier certificate convention as recalled in Section II.

Definition 5 (Practical ISSf Barrier Certificate on a Tube): A continuous function $B : \Xi \to \mathbb{R}$ is a *practical ISSf barrier certificate on* Ξ *for inputs in* U_M if it satisfies (11) for some $\psi \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$, and there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ and $\kappa \geq 0$ s.t.

$$-\overline{\alpha}(|x|_{\mathcal{D}}) - \kappa \;\leq\; B(x) \;\leq\; -\underline{\alpha}(|x|_{\mathcal{D}}), \qquad \forall x \in \Xi \setminus \mathcal{D}.$$

This practical variant relaxes the radial bounds by an additive bias κ. This is the form naturally produced by bounded value-function converse constructions and is sufficient for tube-level safety reasoning on $\Xi \setminus \mathcal{D}$, even though it may not enforce $B(x) \to 0$ as $|x|_{\mathcal{D}} \to 0$.

Remark 1: The bounds (10) make $-B$, $\mathcal{K}_\infty$-equivalent to the safety margin $|x|_{\mathcal{D}}$ on $\Xi \setminus \mathcal{D}$. The dissipation inequality (11) is an ISS-type one-step condition with safety sign convention. When $v = 0$, B decreases (becomes more negative), pushing trajectories away from $\mathcal{D}$ with the term $\rho(\|v\|)$ quantifies the per-step loss of this evolution.

The next proposition shows that an ISSf barrier on Ξ implies the local ISSf (9) for trajectories that remain in $\Xi \setminus \mathcal{D}$ under bounded inputs satisfying a compatibility condition.

Proposition 1 (ISSf Barrier Implies Local ISSf Under Compatibility Condition on Bounded Input): Let $\Xi \subset \mathcal{X}$ be closed with $\mathcal{D} \subset \Xi$ and $r_{\partial\Xi} > 0$. Assume B is an ISSf barrier on Ξ for inputs in U_M in the sense of Definition 4, with associated $\underline{\alpha}, \overline{\alpha}, \psi \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$. Fix $\theta \in (0,1)$ and define

$$\phi(r) := \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \psi^{-1}\left(\frac{\rho(r)}{\theta}\right), \qquad r \geq 0. \qquad (12)$$

Let x be any trajectory of (1) with $x_0 \in \Xi \backslash \mathcal{D}$ and $u \in \mathcal{U}_M$, and let $\tau_{\Xi\backslash\mathcal{D}} := \inf\{k \geq 0 : \; x_k \notin \Xi \setminus \mathcal{D}\}$. Assume the compatibility condition

$$|x_j|_{\mathcal{D}} \geq \phi(\|u_j\|), \forall j \in \mathbb{Z}_{\geq 0} \text{ s.t. } j+1 < \tau_{\Xi\backslash\mathcal{D}}, \qquad (13)$$

holds. Then, for every $k \in \mathbb{Z}_{\geq 1}$ with $k < \tau_{\Xi\backslash\mathcal{D}}$, the trajectory satisfies the local ISSf estimate (9) with $\sigma(s) = s$, $\delta := r_{\partial\Xi}$ with the same input gain ϕ as in (12), and with

$$\mu(s,k) := \overline{\alpha}^{-1}\big(h^k(\underline{\alpha}(s))\big), \qquad h(r) := r + (1-\theta)\psi(\overline{\alpha}^{-1}(r)),$$

where h^k denotes the k-fold iterate of h. In particular, $\mu \in \mathcal{KK}$.

Proof. Let $\widetilde{B} := -B \geq 0$ on $\Xi \setminus \mathcal{D}$, and set $\widetilde{B}_k := \widetilde{B}(x_k)$. From (10), $\underline{\alpha}(|x_k|_{\mathcal{D}}) \leq \widetilde{B}_k \leq \overline{\alpha}(|x_k|_{\mathcal{D}})$, hence $|x_k|_{\mathcal{D}} \geq \overline{\alpha}^{-1}(\widetilde{B}_k)$. Fix $k \in \mathbb{Z}_{\geq 1}$ with $k < \tau_{\Xi\backslash\mathcal{D}}$. For each $j = 0, \ldots, k-1$, both x_j and x_{j+1} belong to $\Xi \setminus \mathcal{D}$. Hence (11) gives $\widetilde{B}_{j+1} - \widetilde{B}_j \geq \psi(|x_j|_{\mathcal{D}}) - \rho(\|u_j\|) \geq \psi(\overline{\alpha}^{-1}(\widetilde{B}_j)) - \rho(\|u_j\|)$. By the compatibility condition and the definition of ϕ, $\underline{\alpha}(|x_j|_{\mathcal{D}}) \geq \overline{\alpha}\Big(\psi^{-1}\Big(\frac{\rho(\|u_j\|)}{\theta}\Big)\Big)$. Since $\widetilde{B}_j \geq \underline{\alpha}(|x_j|_{\mathcal{D}})$, monotonicity of $\overline{\alpha}^{-1}$ and ψ gives $\rho(\|u_j\|) \leq \theta\psi(\overline{\alpha}^{-1}(\widetilde{B}_j))$. Therefore $\widetilde{B}_{j+1} \geq \widetilde{B}_j + (1-\theta)\psi(\overline{\alpha}^{-1}(\widetilde{B}_j)) =: h(\widetilde{B}_j)$. Iterating for $j = 0, \ldots, k-1$ yields $\widetilde{B}_k \geq h^k(\widetilde{B}_0) \geq h^k(\underline{\alpha}(|x_0|_{\mathcal{D}}))$. Thus $|x_k|_{\mathcal{D}} \geq \overline{\alpha}^{-1}(\widetilde{B}_k) \geq \overline{\alpha}^{-1}\big(h^k(\underline{\alpha}(|x_0|_{\mathcal{D}}))\big) = \mu(|x_0|_{\mathcal{D}}, k)$. Since $\mu(|x_0|_{\mathcal{D}}, k) \geq \min\{\mu(|x_0|_{\mathcal{D}}, k), \delta\} - \phi(\|u_{k-1}\|)$, we obtain (9) for all $k \in \mathbb{Z}_{\geq 1}$ with $k < \tau_{\Xi\backslash\mathcal{D}}$. □

Thus, the condition (13) is a sufficient admissibility condition ensuring that the input term in (11) does not dominate the barrier dissipation. The violation of this condition *does not* imply that the system becomes unsafe; it means that the certificate can no longer be used to guarantee the system's safety.

V. Global pISSf Sufficiency

Proposition 1 yields a tube-local ISSf estimate under a pointwise compatibility condition. To obtain a global trajectory-level robustness bound of the form (3), we need the following power-type comparison and dissipation bounds on B.

Assumption 1 (ISSf Barrier Bounds): There exist a function $B : \mathcal{X} \to \mathbb{R}$, constants $c_1, c_2, c_3, c_4 > 0$, exponents $p, q > 0$, and $\kappa \geq 0$. Without loss of generality, consider $c_2 \leq c_1$; then, for all $x \in \mathcal{X}$ and $v \in \mathbb{U}$,

$$-c_1\,|x|_{\mathcal{D}}^p - \kappa \;\leq\; B(x) \;\leq\; -c_2\,|x|_{\mathcal{D}}^p, \qquad (14)$$

$$B\big(F(x,v)\big) - B(x) \;\leq\; -c_3\,|x|_{\mathcal{D}}^p + c_4\,\|v\|^q. \qquad (15)$$

These inequalities induce a linear recursion in $|x_k|_{\mathcal{D}}^p$, which can be unrolled via a discrete-time Bellman–Grönwall (reverse Grönwall) argument to obtain an explicit geometric estimate. A standard result is recalled in the following lemma.

Lemma 2 (Reverse Bellman–Grönwall [2]): Let $a > 1$ and suppose $s_{k+1} \geq a s_k - v_k$ for all $k \geq 0$, with $s_k, v_k \geq 0$. Then

$$s_k \;\geq\; a^k s_0 \;-\; \frac{a^k - 1}{a - 1}\max_{0 \leq i < k} v_i, \qquad \forall k \geq 1.$$

Finally, we characterize the ISSf barrier function and provide sufficiency statement.

Theorem 1: (Global pISSf Sufficiency; Explicit Geometric Bound): Under Assumption 1, set $A := 1 + \frac{c_3}{c_1} > 1$, $\tilde{b} := \frac{c_4}{c_3}$, $\tilde{d} := \frac{\kappa}{c_1}$. Then for all $k \in \mathbb{Z}_{\geq 0}$,

$$|x_k|_{\mathcal{D}}^p \;\geq\; \frac{c_2}{c_1} A^k\,|x_0|_{\mathcal{D}}^p \;-\; (A^k - 1)\,\tilde{b}\,\|u\|_{\infty,[0,k)}^q \;-\; A^k\,\tilde{d}. \qquad (16)$$

Hence (3) holds with $\sigma(s) = s^p$, $\alpha(s,k) = \frac{c_2}{c_1}A^k s^p$, $\phi(r,k) = (A^k - 1)\frac{c_4}{c_3}r^q$, $\gamma_k = A^k \frac{\kappa}{c_1}$.

Proof. Let $B_k := B(x_k)$ and $s_k := |x_k|^p_{\mathcal{D}}$. Since $x_{k+1} = F(x_k, u_k)$, from (14), $s_k \geq (-B_k - \kappa)/c_1$. Substituting into (15) gives $B_{k+1} \leq B_k - c_3\frac{-B_k-\kappa}{c_1} + c_4\|u_k\|^q = AB_k + (A-1)\kappa + c_4\|u_k\|^q$, where $A := 1 + \frac{c_3}{c_1} > 1$. Unrolling this recursion and using Lemma 2 yields $B_k \leq A^k B_0 + \frac{A^k-1}{A-1}\Big((A-1)\kappa + c_4\|u\|^q_{\infty,[0,k)}\Big)$. Using $B_0 \leq -c_2 s_0$ and $B_k \geq -c_1 s_k - \kappa$, we obtain (16) after rearranging and noting that $(A-1) = c_3/c_1$. □

The characterization (16) has the expected ISS structure. Next, we investigate the converse properties.

VI. Converse (Necessity): Constructing an ISSf Barrier on a Tube

We now establish local necessity results strong local ISSf on a compact tube Ξ under bounded inputs implies existence of a (practical) ISSf barrier on Ξ.

Theorem 2 (Converse: Practical ISSf Barrier From Strong Local ISSf): Let $\Xi \subset \mathcal{X}$ be compact with $\mathcal{D} \subset \mathrm{int}(\Xi)$ and $M > 0$. Let $U_M := \{v \in \mathbb{U} : \|v\| \leq M\}$ and $\mathcal{U}_M$ be as in (6). Assume the system is strongly ISSf locally in Ξ w.r.t. $\mathcal{D}$ for inputs $u \in \mathcal{U}_M$ in sense of Definition 2. Let $\bar{T}_\mu(\cdot)$ be the uniform finite exit-time as in (7), define $\ell(x) := \sigma(|x|_{\mathcal{D}})$ on Ξ and, for $x \in \Xi \setminus \mathcal{D}$,

$$S(x) := \sup_{u \in \mathcal{U}_M} \sum_{j=0}^{\tau(x,u)-1} \ell(x_j), \tag{17}$$

$$B(x) := -\frac{1}{1 + S(x)} \in (-1, 0), \tag{18}$$

where $x_0 = x$, $x_{j+1} = F(x_j, u_j)$, and $\tau(x,u) := \tau_{\Xi\setminus\mathcal{D}}(x,u)$. Let $r_\Xi := \max_{x\in\Xi}|x|_{\mathcal{D}}$ and $\ell_{\max} := \max_{x\in\Xi}\ell(x) = \sigma(r_\Xi)$. Then for all $x \in \Xi \setminus \mathcal{D}$ and all $v \in U_M$ such that $x^+ := F(x,v) \in \Xi \setminus \mathcal{D}$,

$$B(x^+) - B(x) \;\leq\; -\Delta_B(|x|_{\mathcal{D}}), \tag{19}$$

$$\Delta_B(s) := \frac{\sigma(s)}{\big(1 + \bar{T}_\mu(s)\,\sigma(r_\Xi)\big)^2}, \tag{20}$$

with the convention $\Delta_B(0) := 0$. Moreover, there exists $\psi \in \mathcal{K}_\infty$ with $\psi(s) \leq \Delta_B(s)$ on $[0, r_\Xi]$ such that $\forall x \in \Xi \setminus \mathcal{D}$, $\forall v \in U_M$, $F(x,v) \in \Xi \setminus \mathcal{D}$:

$$B(x^+) - B(x) \;\leq\; -\psi(|x|_{\mathcal{D}}).$$

Finally, B satisfies the radial and one-step practical ISSf-barrier inequalities on $\Xi\setminus\mathcal{D}$ for inputs in U_M. If B is continuous on $\Xi \setminus \mathcal{D}$ and admits a continuous extension to Ξ, then this extension is a practical ISSf barrier certificate in the sense of Definition 5.

Proof. The proof has four steps: finiteness of S, a Bellman-type inequality for S, one-step decay of B, and strictification together with practical radial bounds. First we will establish the finiteness of S. Since Ξ is compact, ℓ is bounded on Ξ and $\ell_{\max} < \infty$. By Lemma 1, for any $x \in \Xi \setminus \mathcal{D}$ and any $u \in \mathcal{U}_M$, the corresponding exit time is uniformly bounded as $\tau(x,u) \leq \bar{T}_\mu(|x|_{\mathcal{D}})$. Since ℓ is bounded on the compact tube Ξ, this immediately implies

$$\sum_{j=0}^{\tau(x,u)-1} \ell(x_j) \;\leq\; \bar{T}_\mu(|x|_{\mathcal{D}})\,\ell_{\max} < \infty. \tag{21}$$

Hence the worst-case exit-sum functional $S(x)$ is well-defined and finite for every $x \in \Xi \setminus \mathcal{D}$. Taking the supremum yields $S(x) < \infty$ and $B(x) \in (-1, 0)$. Having established finiteness of S, we next exploit its recursive structure and apply shift inequality for S. To relate the value of S at x to its value at the one-step successor x^+, we fix the first input $v \in U_M$ and view the remaining evolution as a continuation problem starting from x^+. For any continuation signal $u^+ \in \mathcal{U}_M$, let $u := (v, u^+) \in \mathcal{U}_M$ be the concatenated input. Since $x^+ = x^+ = F(x,v) \in \Xi \setminus \mathcal{D}$ the corresponding trajectory from x accumulates the immediate stage cost $\ell(x)$ at the first step and then follows the trajectory generated from x^+ under u^+. Consequently, the total exit-sum from x splits into the first-stage contribution plus the residual exit-sum from x^+ i.e., $\tau(x,u) = 1 + \tau(x^+, u^+)$ and

$$\sum_{j=0}^{\tau(x,u)-1} \ell(x_j) = \ell(x) + \sum_{j=0}^{\tau(x^+,u^+)-1} \ell(\hat{x}_j),$$

where $\hat{x}_0 = x^+$ and $\hat{x}_{j+1} = F(\hat{x}_j, u_j^+)$. Since $S(x)$ is defined as the supremum over all admissible inputs from x, it is in particular no smaller than the value obtained by any prescribed first move v followed by an arbitrary continuation u^+. Taking the supremum over all such continuations yields the Bellman-type inequality: $S(x) \geq \ell(x) + S(x^+)$, i.e.

$$S(x^+) - S(x) \leq -\ell(x) \tag{22}$$

This recursive inequality can now be transferred to the transformed functional $B(x) = -\frac{1}{1+S(x)}$. Since $S(x) \geq \ell(x) + S(x^+)$, we have $S(x^+) \leq S(x)$, and therefore $(1 + S(x))(1 + S(x^+)) \leq (1 + S(x))^2$. Using

$$B(x^+) - B(x) = \frac{S(x^+) - S(x)}{(1 + S(x))(1 + S(x^+))}$$

and the shift inequality 22 above, we get

$$B(x^+) - B(x) \;\leq\; -\frac{\ell(x)}{(1 + S(x))^2}. \tag{23}$$

Since $S(x) \leq \bar{T}_\mu(|x|_{\mathcal{D}})\,\ell_{\max}$ and $\ell(x) = \sigma(|x|_{\mathcal{D}})$, this yields

$$B(x^+) - B(x) \;\leq\; -\frac{\sigma(|x|_{\mathcal{D}})}{(1 + \bar{T}_\mu(|x|_{\mathcal{D}})\,\sigma(r_\Xi))^2} \;=\; -\Delta_B(|x|_{\mathcal{D}}).$$

Next, we establish a strict $\mathcal{K}_\infty$ decrease rate. On $[0, r_\Xi]$, $\Delta_B(0) = 0$ and $\Delta_B(s) > 0$ for $s > 0$. Let $p(s) := \inf_{r\in[s,r_\Xi]} \Delta_B(r)$ for $s \in (0, r_\Xi]$ and set $p(0) := 0$. Although Δ_B need not be continuous or monotone, the envelope p is non-decreasing by construction. Moreover, $p(s) > 0$ for every $s > 0$. Indeed, for any fixed $s \in (0, r_\Xi]$ and any $r \in [s, r_\Xi]$, monotonicity of $\bar{T}_\mu$ gives $\bar{T}_\mu(r) \leq \bar{T}_\mu(s) < \infty$, while $\sigma(r) \geq \sigma(s) > 0$. Hence

$$\Delta_B(r) = \frac{\sigma(r)}{\big(1 + \bar{T}_\mu(r)\sigma(r_\Xi)\big)^2} \geq \frac{\sigma(s)}{\big(1 + \bar{T}_\mu(s)\sigma(r_\Xi)\big)^2} > 0,$$

and therefore $p(s) \geq \frac{\sigma(s)}{\left(1+\bar{T}_\mu(s)\sigma(r_\Xi)\right)^2} > 0$. By strictification (see Lemma 3), there exists $\psi \in \mathcal{K}$ such that $\psi(s) \leq p(s) \leq \Delta_B(s)$, $s \in [0, r_\Xi]$. Extending ψ to $\mathcal{K}_\infty$ gives the strict decrease estimate $B(x^+) - B(x) \leq -\psi(|x|_{\mathcal{D}})$.

Next, we establish practical radial bounds. Now, $S(x) \geq 0$ implies $B(x) \in [-1, 0)$, hence $B(x) \geq -1 \geq -|x|_{\mathcal{D}} - 1$, so the lower practical bound holds with $\overline{\alpha}(s) = s$ and $\kappa = 1$. Also, from above, $-B(x) = \frac{1}{1+S(x)} \geq \frac{1}{1+\bar{T}_\mu(|x|_\mathcal{D})\sigma(r_\Xi)}$. Applying again the strictification (Lemma 3) on $[0, r_\Xi]$ yields $\underline{\alpha} \in \mathcal{K}_\infty$ such that $B(x) \leq -\underline{\alpha}(|x|_\mathcal{D})$ on $\Xi \setminus \mathcal{D}$. Finally, since $B(x^+) - B(x) \leq -\psi(|x|_\mathcal{D})$, Definition 5 holds with any $\rho \in \mathcal{K}$ (e.g. $\rho(r) = r$) as $-\psi(|x|_\mathcal{D}) \leq -\psi(|x|_\mathcal{D}) + \rho(\|v\|)$ for all $v \in U_M$. Thus, B satisfies the practical ISSf-barrier inequalities on $\Xi \setminus \mathcal{D}$ for inputs in U_M. If, in addition, B is continuous on Ξ or admits a continuous extension to Ξ, then B is a practical ISSf barrier certificate in the sense of Definition 5. □

Remark 2 (Regularity of the Constructed Barrier): Assume F is continuous and U_M is compact. On any compact set $K_\varepsilon := \{x \in \Xi : |x|_\mathcal{D} \geq \varepsilon\} \subset \Xi \setminus \mathcal{D}$, Lemma 1 provides a uniform bound $\tau(x, u) \leq N_\varepsilon$ for all $x \in K_\varepsilon$ and $u \in \mathcal{U}_M$. Consequently, the value functional S is finite on K_ε. Upper semi-continuity of S follows from standard dynamic-programming / maximum-theorem arguments on finite-horizon bounded-input costs. Continuity holds under additional regularity assumptions on the exit event (e.g. no-grazing conditions).

Corollary 1 (Smooth Barrier Away From $\mathcal{D}$): Assume B is continuous on a neighborhood of K_ε. Given Theorem 2, fix any $\varepsilon > 0$ and define the compact set $K_\varepsilon := \{x \in \Xi : |x|_\mathcal{D} \geq \varepsilon\} \subset \Xi \setminus \mathcal{D}$. Then, following standard mollification strategy [5], there exists an open set Ω with $K_\varepsilon \subset \Omega$ and, for any $\eta \in (0, 1/2)$, a function $\widetilde{B} \in C^\infty(\Omega)$ such that $\sup_{x \in K_\varepsilon} |\widetilde{B}(x) - B(x)| \leq \eta\,\psi(\varepsilon)$ and $\widetilde{B}(F(x, v)) - \widetilde{B}(x) \leq -(1 - 2\eta)\,\psi(|x|_\mathcal{D})$ for all $(x, v) \in K_\varepsilon \times U_M$ with $F(x, v) \in K_\varepsilon$.

Discussion: In particular, strong local ISSf on Ξ implies a uniform finite-exit bound (Lemma 1), and therefore Theorem 2 yields a practical tube ISSf barrier. A directly checkable sufficient condition for Definition 2 is robust expansion of the safety margin on the tube. For example, if there exist $q > 1$, $\sigma \in \mathcal{K}$, and $\phi \in \mathcal{K}$ such that $\sigma(|F(x, v)|_\mathcal{D}) \geq q\,\sigma(|x|_\mathcal{D})$ for all $x \in \Xi \setminus \mathcal{D}$ and $v \in U_M$ with $F(x, v) \in \Xi \setminus \mathcal{D}$, then strong local ISSf holds with $\mu(s, k) = q^k \sigma(s)$ and $\bar{T}_\mu(s) = \min\{k \geq 1 : q^k \sigma(s) > \sigma(r_\Xi) + \phi(M)\}$. Thus Theorem 2 can be invoked by checking a one-step robust margin-expansion inequality on the compact tube (see Example 1). The resulting exit-sum S is finite-horizon by Lemma 1, so it can be approximated by dynamic programming (for example, grid based), or scenario-based maximization on Ξ.

VII. Conclusion

This letter developed an input-to-state safety framework for general discrete-time nonlinear systems with respect to a prescribed unsafe set. We introduced a global practical ISSf notion and two tube-local notions, namely finite-threshold strong local ISSf and saturated local ISSf, together with ISSf barrier certificates on compact tubes. We showed that a tube ISSf barrier implies a local ISSf estimate under an explicit state-input compatibility condition, and that global power-type comparison and dissipation bounds yield an explicit practical ISSf bound. The main contribution is a tube-local converse construction: finite-threshold strong local ISSf under bounded inputs yields a worst-case exit-sum function satisfying practical ISSf-barrier inequalities, and under additional regularity this function defines a practical tube ISSf barrier certificate. Future work will focus on synthesis by enforcing robust margin-expansion or ISSf-barrier inequalities through MPC/QP safety filters, numerical approximation of the exit-sum barrier on compact tubes, converse results under weaker local ISSf assumptions, and interconnections.

Appendix

Lemma 3 ([7]): Let $a > 0$ and let $p : [0, a] \to \mathbb{R}_{\geq 0}$ be non-decreasing with $p(s) > 0$ for all $s \in (0, a]$. Define $\alpha : [0, a] \to \mathbb{R}_{\geq 0}$ by $\alpha(s) := \frac{1}{a}\int_0^s p(r)\,dr$. Then $\alpha \in \mathcal{K}$ on $[0, a]$ and $\alpha(s) \leq p(s)$ for all $s \in [0, a]$. Moreover, α admits an extension $\tilde{\alpha} \in \mathcal{K}_\infty$.

Proof. Since $p(r) > 0$ for $r > 0$, α is strictly increasing and $\alpha(0) = 0$. As p is non-decreasing, $\alpha(s) \leq \frac{1}{a}\int_0^s p(s)\,dr = \frac{s}{a}p(s) \leq p(s)$ on $[0, a]$. Extend by $\tilde{\alpha}(s) = \alpha(s)$ for $s \leq a$ and $\tilde{\alpha}(s) = \alpha(a) + (s - a)$ for $s \geq a$. □

References

[1] A.D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 3861–3876, Aug. 2017.

[2] L. Grune and C. M. Kellett, "ISS-Lyapunov functions for discontinuous discrete-time systems," *IEEE Trans. Autom. Control*, vol. 59, no. 11, pp. 3098–3103, Nov. 2014.

[3] M. S. Jha and B. Kiumarsi, "Off-policy safe reinforcement learning for nonlinear discrete-time systems," *Neurocomputing*, vol. 611, Jan. 2025, Art. no. 128677.

[4] M. S. Jha, S. Marthi, K. G. Vamvoudakis, S. Kanso, and D. Theilliol, "Input-to-state safety for reinforcement learning," *IEEE Trans. Neural Netw. Learn. Syst.*, pp. 1–15, 2026.

[5] Z.-P. Jiang and Y. Wang, "A converse Lyapunov theorem for discrete-time systems with disturbances," *Syst. Control Lett.*, vol. 45, no. 1, pp. 49–58, Jan. 2002.

[6] S. Kolathaya and A. D. Ames, "Input-to-state safety with control barrier functions," *IEEE Control Syst. Lett.*, vol. 3, no. 1, pp. 108–113, Jan. 2019.

[7] H. Khalil, "Nonlinear systems," *Tech. Rep.*, 2002.

[8] S. Kanso, M. S. Jha, and D. Theilliol, "Off-policy model-based end-to-end safe reinforcement learning," *Int. J. Robust Nonlinear Control*, vol. 34, no. 4, pp. 2806–2831, Mar. 2024.

[9] S. Prajna and A. Jadbabaie, "Safety verification of hybrid systems using barrier certificates," in *Proc. Int. Workshop Hybrid Syst., Comput. Control. Cham*, Switzerland: Springer, 2004, pp. 477–492.

[10] M. Z. Romdlony and B. Jayawardhana, "Robustness analysis of systems' safety through a new notion of input-to-state safety," *Int. J. Robust Nonlinear Control*, vol. 29, no. 7, pp. 2125–2136, May 2019.

[11] M. Z. Romdlony and B. Jayawardhana, "Stabilization with guaranteed safety using control Lyapunov–barrier function," *Automatica*, vol. 66, pp. 39–47, Apr. 2016.

[12] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[13] P. Wieland and F. Allgöwer, "Constructive safety using control barrier functions," *IFAC Proc. Volumes*, vol. 40, no. 12, pp. 462–467, 2007.