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DEGRADATION TOLERANT OPTIMAL CONTROL DESIGN FOR STOCHASTIC LINEAR SYSTEMS

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Safety-critical and mission-critical systems are often sensitive to functional degradation at the system or component level. Such degradation dynamics are often dependent on system usage (or control input) and can lead to significant loss and potential system failure. As such, it becomes imperative to develop control designs that are able to ensure system stability and performance, whilst mitigating the effects of incipient degradation by modulating the control input appropriately. In this context, this paper proposes a novel approach based on an optimal control theory framework wherein the degradation state of the system is considered in the augmented system model, and estimated using sensor measurements. Further, it is incorporated within the optimal control paradigm leading to control law that results in deceleration of degradation rate at the cost of system performance whilst ensuring system stability. To that end, the speed of degradation and the state of the system in discrete time are considered to develop a linear quadratic tracker (LQT) and regulator (LQR) over a finite horizon in a mathematically rigorous manner. Simulation studies are performed to assess the proposed approach.

Keywords: Linear Quadratic Gaussian Control, Optimal Control, Kalman Filter, Stochastic Linear System, Degradation.

1. Introduction

Traditional control system designs (Stengel, 1986), (Åström and Wittenmark, 1995) focus only on the stability and the performance without taking into consideration the effects of aging, fatigue, and damage of the concerned components and without minimizing the risk of failure.
However, safety-critical systems (Knight, 2002) arise in several application areas, such as transportation and air-traffic control systems, space systems, nuclear plants and automated industrial processes. The evolution of such complex systems call for development of new control technologies that maintain system stability and performance specifications, and also address the progressive incipient degradation.

In this context, recent works include approaches such as adaptive or robust control to address issues where the degree of failure may be unknown. In (Bole *et al.*, 2010), a fault adaptive control is proposed for incipient fault modes growing to catastrophic failure conditions. The methodology is developed for a finite constrained optimization problem where the model of the system and the degradation is supposed to be known. (Zhang *et al.*, 2022) develops a reconfiguration control method using a multiple-model based adaptive control. The proposed control law allows to handle component faults while maintaining the performance of electro-hydraulic position servo system. Moreover, fault tolerant control design (Noura *et al.*, 2009), (Blanke *et al.*, 2006) has been developed for various industrial, mission critical and safety critical systems that operate in closed loop, in order to compensate for fault occurrence. In (Hamdi *et al.*, 2021), a fault tolerant control was introduced for delayed linear parameter varying systems including disturbances and actuator faults.

Very recently, new methods are developed such that useful life of critical systems can be developed. In this context, health aware control has recently become one of the domains where control law is designed, taking into account the state of health (SoH) and/or Remaining Useful Life (RUL) prognostics of critical components. Some prominent works have proposed methods to develop control laws that attempt to extend the RUL of component/system such as (Lipiec *et al.*, 2022), (Pour *et al.*, 2021), (Rodriguez *et al.*, 2018), (Salazar *et al.*, 2017). Also, in the framework of model predictive control (MPC), several works were adopted to design a controller that ensures robustness to particular failures, thus reducing their impact on the system (Brown *et al.*, 2010), (Brown *et al.*, 2021). When all the states of a system are not measurable, it can be challenging to design effective controllers that can maintain good performance in the presence of these uncertainties.There are several techniques that can be assigned to estimate the states of systems and the degradation state such as Kalman Filter (Durrant-Whyte, 2006) , Extended Kalman Filter (Kanso *et al.*, 2022), (Obando *et al.*, 2021), (Bressel *et al.*, 2016). Particle filtering (Jha *et al.*, 2016), etc. In the framework of linear system, Kalman filtering is often used for real-time control applications due to its low computational complexity and convergence guarantee. Combining Kalman Filter and linear quadratic control (LQC) yields to Linear Quadratic Gaussian (LQG) control (Lewis *et al.*, 2012), (Söderström, 2002). LQG control is used to optimize the performance of linear systems in the presence of additive white Gaussian noise. It is widely used in a variety of applications to maintain good control performance in the presence of noise (Athans, 1971). The Kalman filter is used to estimate the state of the system based on noisy measurements, and the estimate of the state is used to compute the optimal control input.

Incipient degradation is usually a slow-dynamic phenomenon, hidden and often not directly measurable using sensors. As such, incorporation of degradation state within control design is a non-trivial task (Söderström, 2002). Another particular challenge that arises is that the degradation states may not be measurable (Félix *et al.*, 2022). Thus, it can be difficult to accurately assess the extent of degradation and to design controllers that can adapt to changing degradation levels over time or decelerate its speed.
Most of the existing work take into account fault tolerance within the control design without addressing the incipient functional degradation phenomena that leads to such faults and consequently, system failure. On the other hand, very few works have addressed the problem emanating due to degradation that is often not measurable and incipient in nature. In this context, this paper proposes a novel approach based in optimal control theory framework wherein the degradation state of the system is considered in the augmented system model, and estimated using sensor measurements. Further, it is incorporated within optimal control paradigm leading to control law that results in deceleration of degradation rate at the cost of system performance whilst insuring system stability.

This works is an extension of the previous work (Kanso *et al.*, 2023),wherein a linear quadratic regulator (LQR) and tracker (LQT) was designed for deterministic discrete-time linear system in the presence of a linear degradation, and the full information of the state and the degradation was considered available.
This paper aims to extend the previous work and address the cases of incomplete states information thereby addressing the stochastic systems using LQG control. The main scientific contribution is the proposition of a novel degradation tolerant approach based on optimal control theory for a stochastic discrete-time linear system with partially measurable states and degradation.

This paper is organized as follows. Section 2 introduces the problem statement. Section 3 presents the proposed reconfiguration approach for deterministic systems. Section 4 develops the LQG control for incomplete state information. Section 5 examines the feasibility of the proposed approach using an academic example. Finally, the conclusion summarizes the significant advances and presents the future perspectives.

2. Problem Formulation

The degradation of system's components affects the performance and the stability of the system. The state of degradation or deterioration, considered as a health indicator, as well affects directly the remaining useful life of the active system, consequently reducing the usability and the productivity of the system. Moreover, the state of health (SoH) is is predominantly influenced by the states of the system, and implicitly affected by the action of the controller. As such, development of an optimal approach for performing a control action that takes into account the performance requirements, the stability and also the SoH of the system gains paramount importance for such systems undergoing components degradation.

This paper focuses on linear MIMO (Multiple Inputs Multiple Outputs) discrete-time systems represented by the state transition, control and observation matrices, $A_1 \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{n\times l}$, $B_1 \in \mathbb{R}^{n\times m}$ and $C_1 \in \mathbb{R}^{p\times n}$.

$$x_{k+1} = A_1 x_k + A_2 d_k + B_1 u_k \tag{1}$$

$$y_k = C_1 x_k \tag{2}$$

where $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $d \in \mathbb{R}^l$ and $y \in \mathbb{R}^p$ correspond respectively to the input, state of the system, state of degradation and measurement vectors. The system is affected by the degradation in an affine manner and the degradation evolution is described by the following state-space representation:

$$d_{k+1} = A_3 x_k + A_4 d_k \tag{3}$$

with $A_3 \in \mathbb{R}^{l\times n}$ and $A_4 \in \mathbb{R}^{l\times l}$. In most cases, the evolution of the degradation is monotonic and irreversible, moreover, it is generally unknown. In this work, the current state of degradation is assumed to be dependent on the previous state of degradation and also the previous state of the system.
In order to maintain the performance of the system while minimizing the energy and the speed of evolution of degradation, a quadratic utility function is defined by:

$$\mathscr{U}_k = (C_1x_k - r_k)^T Q(C_1x_k - r_k) + u_k^T Ru_k + \Delta d_k^T Q_1 \Delta d_k \tag{4}$$

where r_k is the desired reference trajectory and Δd_k is the rate of evolution of degradation described by(5).

$$\Delta d_k = d_{k+1} - d_k = (A_4 - I)d_k + A_3x_k \tag{5}$$

The utility function (4) is used to develop the performance index of a linear quadratic tracker problem, which gives the following quadratic cost function :

$$\begin{aligned} J_0 = &\frac{1}{2}[(C_1x_N - r_N)^T \bar{S}_N(C_1x_N - r_N) + \Delta d_N^T \bar{P}_N \Delta d_N] \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}[(C_1x_k - r_k)^T Q(C_1x_k - r_k) + u_k^T Ru_k \\ &+ \Delta d_k^T Q_1 \Delta d_k] \end{aligned} \tag{6}$$

Q, Q_1, R, $\bar{S}_N$ and $\bar{P}_N$ are symmetric positive definite cost-weighting matrices and $|R| \neq 0$. The initial plant and degradation state are given as x_0 and d_0 respectively.

In the following section, the control problem will be addressed for the case of deterministic system while minimizing the rate of evolution of degradation.

3. Optimal Reconfiguration Control of Deterministic Systems

In this section, an optimal control based approach is developed that allows the synthesis of a state feedback control law using the minimization of a quadratic criterion involving the state, the control and the rate of evolution of degradation. The problem posed is to bring the state to any reference track. This problem is equivalent to bringing the state to the equilibrium (zero) starting from a non-zero initial condition. Hence, in section 3.1, the solution of the problem is developed for a LQT, and then the solution is deduced for a LQR problem. The constructed controller is for deterministic systems with fully measurable states

3.1. Linear Quadratic Tracker . This section synthesizes an optimal control law that forces the system to track a desired reference trajectory r_k over a specified time interval $[0, N]$. The cost function (6) is sensitive to the tracking error, the input, and Δd to force the state to reach the reference and to decelerate the speed of evolution of degradation. Using (5) to eliminate Δd in (6) gives:

$$\begin{aligned} J_0 = &\frac{1}{2}[x_N^T(C^T\bar{S}_NC + A_3^T\bar{P}_NA_3)x_N \\ &+ d_N^T(A4 - I)^T\bar{P}_N(A_4 - I)d_N + r_N^T\bar{S}_Nr_N \\ &- x_N^TC^T\bar{S}_Nr_N - r_N^T\bar{S}_NCx_N \\ &+ d_N^T(A_4 - I)^T\bar{P}_NA_3x_N + x_N^TA_3^T\bar{P}_N(A_4 - I)d_N] \\ &+ \frac{1}{2}\sum_{k=0}^{N-1}[x_k^T(C^TQC + A_3^TQ_1A_3)x_k + u_k^TRu_k \\ &+ d_k^T(A4 - I)^TQ_1(A_4 - I)d_k + r_k^TQr_k \\ &- x_k^TC^TQr_k - r_k^TQCx_k + d_k^T(A_4 - I)^TQ_1A_3x_k \\ &+ x_k^TA_3^TQ_1(A_4 - I)d_k] \end{aligned} \tag{7}$$

To solve the LQT problem, the Hamiltonian is first considered in order to derive the necessary conditions. The Hamiltonian function is defined by the following equation:

$$\begin{aligned} H_k = &\frac{1}{2}[x_k^T(C_1^TQC_1 + A_3^TQ_1A_3)x_k + u_k^TRu_k \\ &+ d_k^T(A4 - I)^TQ_1(A_4 - I)d_k + r_k^TQr_k \\ &- x_k^TC_1^TQr_k - r_k^TQC_1x_k \\ &+ d_k^T(A_4 - I)^TQ_1A_3x_k + x_k^TA_3^TQ_1(A_4 - I)d_k] \\ &+ \lambda_{k+1}[A_1x_k + A_2d_k + B_1u_k] \end{aligned} \tag{8}$$

where $\lambda_k \in \mathbb{R}^n$ is the costate of the system and it's given by:

$$\begin{aligned} \lambda_k &= \frac{\partial H_k}{\partial x_k} \\ &= (C_1^TQC_1 + A_3^TQ_1A_3)x_k + A_1^T\lambda_{k+1} \\ &\quad + A_3^TQ_1(A_4 - I)d_k - C_1^TQr_k \end{aligned} \tag{9}$$

Solving the stationarity condition $\frac{\partial H_k}{\partial u_k} = 0$, yields to:

$$u_k = -R^{-1}B_1^T\lambda_{k+1} \tag{10}$$

If the optimal λ_k can be found, 10 can be used to find the optimal control. Moreover, the boundary condition is given by:

$$\begin{aligned} \lambda_N &= \frac{\partial \Phi_N}{\partial x_N} \\ &= (C_1^T\bar{S}_NC_1 + A_3^T\bar{P}_NA_3)x_N \\ &\quad + A_3^T\bar{P}_N(A_4 - I)d_N - C_1^T\bar{S}_Nr_N \end{aligned} \tag{11}$$

with

$$\Phi_N = \frac{1}{2}[x_N^T(C_1^T\bar{S}_NC_1 + A_3^T\bar{P}_NA_3)x_N + d_N^T(A4-I)^T\bar{P}_N(A_4-I)d_N + r_N^T\bar{S}_Nr_N - x_N^TC_1^T\bar{S}_Nr_N - r_N^T\bar{S}_NC_1x_N + d_N^T(A_4-I)^T\bar{P}_NA_3x_N + x_N^TA_3^T\bar{P}_N(A_4-I)d_N] \quad (12)$$

Thus, assuming that a linear relation like (11) holds for all times $k \leq N$, the costate equation can be written as:

$$\lambda_k = S_kx_k + P_kd_k - q_k \quad (13)$$

Using (13) in the state equation (1) to get:

$$x_{k+1} = (I + B_1R^{-1}B1^TS_{k+1})^{-1}[(A_1 - B_1R^{-1}B_1^TP_{k+1}A_3)x_k + (A_2 - B_1R^{-1}B_1^TP_{k+1}A_4)d_k + B_1R^{-1}B_1^Tq_{k+1}] \quad (14)$$

Using (14) and (13) in the costate equation (9) gives

$$\begin{aligned} S_kx_k + P_kd_k - q_k &= [C_1^TQC_1 + A_3^TQ_1A_3]x_k + A_1^TS_{k+1}[I + B_1R^{-1}B_1^TS_{k+1}]^{-1}[A_1 - B_1R^{-1}B_1^TP_{k+1}A_3]x_k \\ &+ A_1^TS_{k+1}[I + B_1R^{-1}B_1^TS_{k+1}]^{-1}[A_2 - B_1R^{-1}B_1^TP_{k+1}A_4]d_k \\ &+ A_1^TS_{k+1}[I + B_1R^{-1}B_1^TS_{k+1}]^{-1}B_1R^{-1}B_1^Tq_{k+1} \\ &+ A_1^TP_{k+1}A_4d_k + A_1^TP_{k+1}A_3x_k \\ &+ A_3^TQ_1(A_4-I)d_k - A_1q_{k+1} - C_1^TQr_k \end{aligned} \quad (15)$$

This equation must hold for all state sequences x_k and d_k given any x_0 and d_0, leading to:

$$S_k = C_1^TQC_1 + A_3^TQ_1A_3 + A_1^TS_{k+1}(I + B_1R^{-1}B_1^TS_{k+1})^{-1}(A_1 - B_1R^{-1}B_1^TP_{k+1}A_3) + A_1^TP_{k+1}A_3 \quad (16)$$

$$P_k = A_1^TS_{k+1}(I + B_1R^{-1}B_1^TS_{k+1})^{-1}(A_2 - B_1R^{-1}B_1^TP_{k+1}A_4) + A_1^TP_{k+1}A_4 + A_3^TQ_1(A_4-I) \quad (17)$$

$$q_k = A_1^Tq_{k+1} + C_1^TQr_k - A_1^TS_{k+1}(I + B_1R^{-1}B_1^TS_{k+1})^{-1}B_1R^{-1}B_1^Tq_{k+1} \quad (18)$$

By comparing (11) and (13), the boundary conditions for these recursions are:

$$\begin{aligned} S_N &= C_1^T\bar{S}_NC_1 + A_3^T\bar{P}_NA_3 \\ P_N &= A_3^T\bar{P}_N(A_4-I) \\ v_N &= -C_1^T\bar{S}_Nr_N \end{aligned} \quad (19)$$

Since the sequences S_k, P_k and q_k can be computed, assumption (13) is valid, and the optimal control is:

$$u_k = -R^{-1}B_1^T(S_{k+1}x_{k+1} + P_{k+1}d_{k+1} - q_{k+1}) \quad (20)$$

Substituting (1) and (3) in (20) yields to:

$$u_k = -K_k^xx_k - K_k^dd_k + K_k^qq_{k+1} \quad (21)$$

with

$$K_k^x = (R + B_1^TS_{k+1}B_1)^{-1}B^T(S_{k+1}A_1 + P_{k+1}A_3) \quad (22)$$

$$K_k^d = (R + B_1^TS_{k+1}B_1)^{-1}B^T(S_{k+1}A_2 + P_{k+1}A_4) \quad (23)$$

$$K_k^q = (R + B_1^TS_{k+1}B_1)^{-1}B^T \quad (24)$$

Equations (22), (23) and (24) are solved offline and backwards in time, starting from time $N \to 0$.

The solution for the LQR is reached by determining the control sequence $u_0, u_1, ...,u_{N-1}$ that minimizes J_0 in (25).

$$J_0 = \frac{1}{2}(x_N^T\bar{S}_Nx_N + \Delta d_N^T\bar{P}_N\Delta d_N) + \frac{1}{2}\sum_{k=0}^{N-1} x_k^TQx_k + u_k^TRu_k + \Delta d_k^TQ_1\Delta d_k \quad (25)$$

However, the regulation problem is nothing but a tracking problem where the reference is the equilibrium (zero). Thus, in this case $q_N = 0$, which implies that the optimal control takes the form:

$$u_k = -K_k^xx_k - K_k^dd_k \quad (26)$$

where K_k^x and K_k^d are computed using equations (22) and (23) respectively.

The controllers in (21) and (26) are a full state and degradation feedback, so they require a complete information of the states and the degradation. In many real-world systems, it is not possible to measure all of the states directly and especially the degradation state. This can be due to a variety of factors, such as the complexity of the system, the cost or difficulty of obtaining measurements, or the inherent limitations of the measurement devices. As a result, it is often necessary to estimate the states of the system based on partial or noisy measurements, using techniques such as Kalman filtering. In the following section, the control problem of stochastic linear systems with incomplete state information will be addressed.

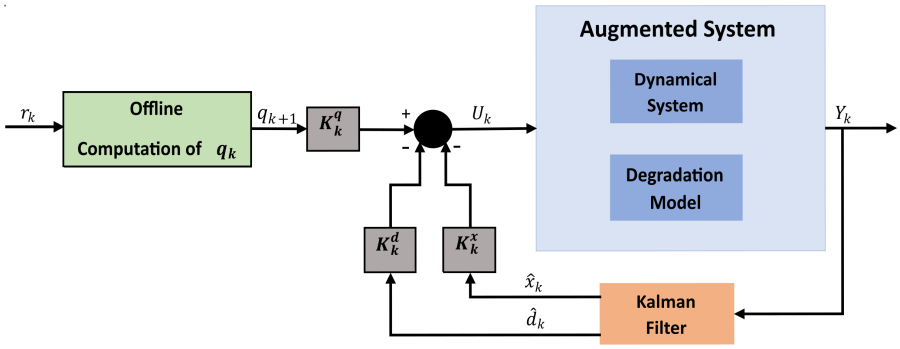


Fig. 1. Regulator design using state feedback and Kalman Filter as observer.

4. Optimal Control with Incomplete State Information

In the previous section, the system was assumed to be is exactly known and that there is no modeling inaccuracies, disturbances, or noises. In control design, all states are not often available for feedback purposes, only measurements are accessible, this can be due to various factors, such as the cost or complexity of measurement, the limitations of available sensors. In this section, an incomplete state information is assumed to be available and the measurements are considered noisy. To solve this problem, Kalman filter observer will be used to estimate the state and the degradation from noisy measurements. LQT control combined with Kalman Filter constitute together the Linear Quadratic Gaussian (LQG) control. This latter provide a powerful method for controlling linear systems in the presence of noise.

Suppose the following systems described by the stochastic dynamical equations:

$$\begin{aligned} x_{k+1} &= A_1 x_k + A_2 d_k + B_1 u_k + w_{1,k} \\ d_{k+1} &= A_3 x_k + A_4 d_k + w_{2,k} \\ y_k &= C_1 x_k + v_k \end{aligned} \tag{27}$$

The signals $w_{1,k}$ and $w_{2,k}$ are unknown process noise that acts to disturb respectively the dynamical system and the degradation, and it could represent unmodeled high-frequency plant dynamics,or the effects of wind gusts for instance. The signal v_k is an unknown measurement noise that impair the measurements and it represent sensor's noise. The signals $w_{1,k}$, $w_{2,k}$ and v_k are uncorrelated.
Consider the following augmented system composed of the dynamical system and the degradation with $X_k = \begin{bmatrix} x_k & d_k \end{bmatrix}^T \in R^{n+l}$ and $U_k = u_k$:

$$\begin{aligned} X_{k+1} &= AX_k + BU_k + w_k \\ Y_k &= CX_k + v_k \end{aligned} \tag{28}$$

where $A = \begin{bmatrix} A1 & A2 \\ A3 & A4 \end{bmatrix}$, $B = \begin{bmatrix} B1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$ and $w_k = \begin{bmatrix} w_{1,k} \\ w_{2,k} \end{bmatrix}$. $w_k \sim (0, Q_{obs})$, $v_k \sim (0, R_{obs})$ are white noise processes orthogonal to each other.
Suppose that the full state-feedback control:

$$\begin{aligned} u_k &= -K_k^x x_k - K_k^d d_k + K_k^q q_{k+1} \\ &= -K_k^X X_k + K_k^q q_{k+1} \end{aligned} \tag{29}$$

with $K_k^X = \begin{bmatrix} K_k^x & K_k^d \end{bmatrix}$, the same feedback vector is used as when the system was deterministic and the states were known (Stengel, 1986).
The closed-loop system become:

$$X_{k+1} = (A - BK_k^X)X_k + BK_k^q q_{k+1} + w_k. \tag{30}$$

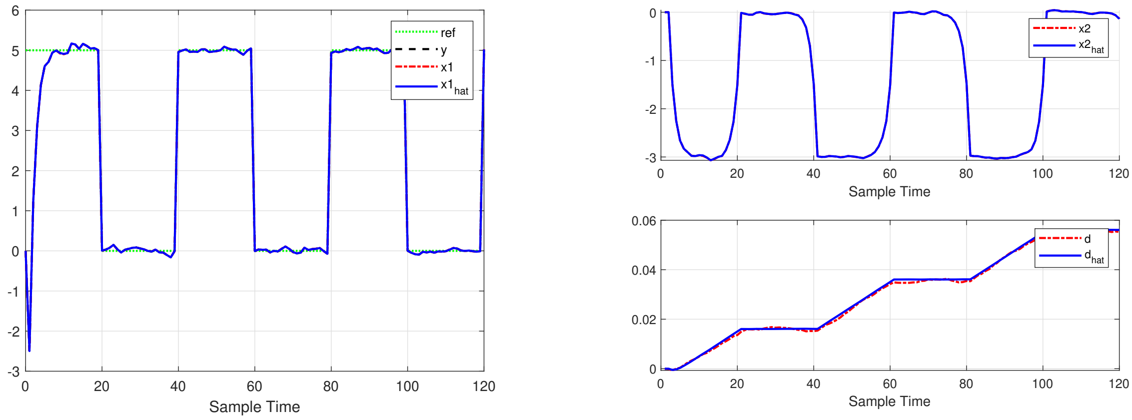
The control law (29) cannot be implemented because not all the states are usually measurable. Now, consider a Kalman filter designed as:

$$\hat{X}_{k+1} = (A - L_{k+1}C)\hat{X}_k + BU_k + L_{k+1}Y_k \tag{31}$$

where the filter gain L_{k+1} is obtained using the Kalman Filter algorithm (Durrant-Whyte, 2006).
The feedback of the estimate $\hat{X}_k$ is used instead of the actual state X_k. Hence, the feedback control law becomes

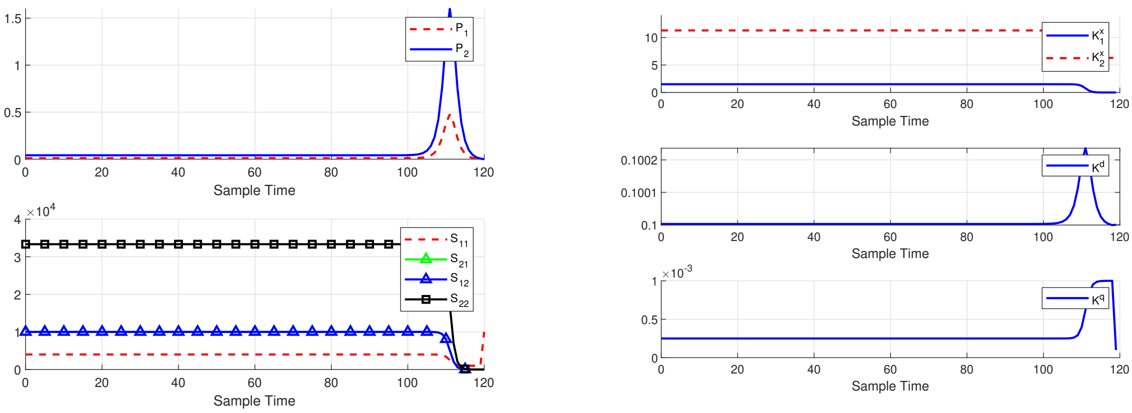
$$u_k = -K_k^X \hat{X}_k + K_k^q q_{k+1} \tag{32}$$

The closed-loop structure using this controller is illustrated in Fig. 1.
The state feedback gains and the observer gain can be developed separately to obtain the desired observer behavior and closed-loop plant behavior. This leads to the separation theorem (Lewis *et al.*, 2012), which is the core of the modern control design. To verify the effectiveness of the developed control schemes, a finite horizon tracker is implemented on an academic example in the next section.



(a) (b)

Fig. 2. Trajectory of estimated and real states and degradation in closed loop: $x_{1,k}$, $\hat{x}_{1,k}$ and measurement y_k (a), the second state $x_{2,k}$, $\hat{x}_{2,k}$ and the degradation d_k and $\hat{d}_k$ (b).



(a) (b)

Fig. 3. Convergence of S_k, P_k , K_k^x, K_k^d and K_k^q: evolution of matrices P_k and S_k with respect to time (a), evolution of the controller gains K_k^x, K_k^d and K_k^q with respect to time (b).

5. Simulation Results

Consider the following unstable stochastic discrete-time linear system :

$$x_{k+1} = \begin{bmatrix} 0 & 6.3 \\ 0.6 & 2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + 0.1 d_k + w_{1,k}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k$$

The dynamic of evolution of degradation is described by the following equation:

$$d_{k+1} = \begin{bmatrix} 2 \times 10^{-3} & 0 \end{bmatrix} x_k + d_k + w_{2,k}$$

The weighting matrices of the tracker are chosen as:

$$Q = \begin{bmatrix} 10^3 & 0 \\ 0 & 10^3 \end{bmatrix}; Q_1 = 10^3; R = 0.01$$
$$\text{and } \bar{S}_N = Q \,; \bar{P}_N = Q_1$$

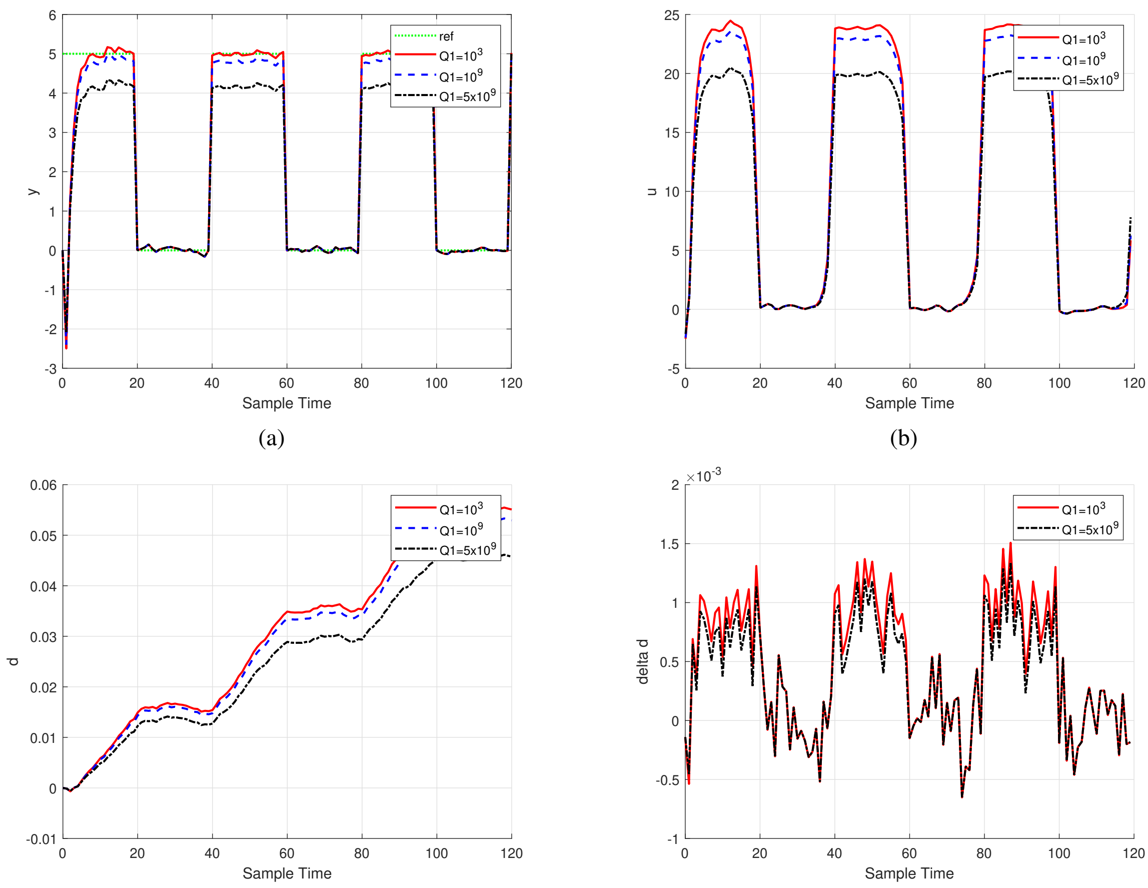
The augmented system is given by:

$$X_{k+1} = \begin{bmatrix} 0 & 6.3 & 0.1 \\ 0.6 & 2 & 0 \\ 2 \times 10^3 & 0 & 1 \end{bmatrix} X_k + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U_k + w_k$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X_k + v_k$$

The first step in the Kalman Filter algorithm is to initialize the states and to adjust the covariance matrices to make the filter work properly. The noises w_k and v_k are assumed to be Gaussian with zero mean and variances Q_{obs} and R_{obs} respectively with:

$$Q_{obs} = \begin{bmatrix} \sigma_{w_1}^2 & 0 & 0 \\ 0 & \sigma_{w_1}^2 & 0 \\ 0 & 0 & \sigma_{w_2}^2 \end{bmatrix} \text{ and } R_{obs} = \sigma_v^2$$



(a) (b) (c) (d)

Fig. 4. Performance of the system and the degradation for different values of Q_1: output of the system (a), control input of the system (b), the degradation (c) and the rate of evolution of degradation (d).

where $\sigma_{w_1} = 6 \times 10^{-4}$, $\sigma_{w_2} = 3 \times 10^{-4}$ and $\sigma_v = 3 \times 10^{-3}$.

Fig. 2a shows the trajectory of $x_{1,k}$, $\hat{x}_{1,k}$ and the measurements y_k for $N = 120$. A strong correlation is observed between the three curves, which implies that the Kalman Filter is able to estimate x_1 since $\hat{x}_1$ and the measured values are overlapped. Moreover, the output y_k tracks the reference r_k. This indicates that the controller is able to force the output y_k to reach the desired trajectory and to stabilise the system while using the estimated state feedback.
Consistent results are obtained in Fig. 2b since the trajectory of $x_{2,k}$ and $\hat{x}_{2,k}$ coincide. The second graph in Fig 2b shows the evolution of the estimated and the real value of the degradation in closed loop, the two curves are correlated with a small error between d_k and $\hat{d}_k$.
Fig. 3a displays the evolution of the matrices P_k and S_k which form respectively the solution of equations (17) and (16). The values are computed offline backward in time from N to 0. It can be seen from these two figures that the matrices parameters converge respectively to P_0 and S_0, for any S_N and P_N, when k approaches to 0.
Similar performance is obtained in Fig. 3b, which is reasonable, as K^x_k,K^d_k and K^q_k are computed with respect to P_{k+1} and S_{k+1} (22-24). As $N \to \infty$, P_k and S_k converge to P_∞ and S_∞ , which implies that K^x_k, K^d_k and K^q_k reach steady-state values K^x_∞, K^d_∞ and K^q_∞.
Thus, in this case, the optimal control can be written as follow:

$$u_k = -K^x_\infty x_k - K^d_\infty d_k + K^q_\infty q_{k+1}$$

A disadvantage of this formulation is that q_k needs to be computed offline using the backward recursion (18).

The main objective of the developed work is to decelerate the speed of evolution of degradation Δd_k. Looking at eq. (6), it can be seen that the weighting matrix Q_1 is strongly affecting the progression of the degradation. Thus, the impact of Q_1 on the system behaviour will be studied in the following results.
For different value of Q_1 =[10^3, 10^9,5×10^9] and for

a finite horizon $N = 120$, the trajectory of the output y_k is represented in Fig. 4a. It can be seen that by increasing the value of Q_1, the steady state error between the output and the reference increases, thus the input u_k reduces in its turn as shown in Fig. 4b. Moreover, Fig. 4c displays the evolution of the degradation for different values of Q_1. It shows that the final value of the degradation d_N decreases when Q_1 increases. This means that the controller tries to obtain a trad-off between the performance and the speed of degradation. As such by augmenting the value of Q_1, the controller will prioritize reducing the rate of evolution of degradation over the performance of the system.
Fig. 4d shows consistent results with the previous one, it displays the rate of evolution of the degradation for different Q_1 and it confirms that the speed of degradation is slower when Q_1 is large.

6. Conclusions and Perspectives

This paper proposes a degradation tolerant control (DTC) design based on LQG approach, where the degradation is hidden. A finite horizon optimization approach is developed for linear systems, where this latter was supposed to be affected by a linear degradation in an affine manner. The degradation and the state of system are considered not fully measurable. Thus, Kalman Filter is employed to estimate the states of the systems and the degradation in order to perform the state feedback control. It allows a fast estimation with a negligible residue, using noisy measurements. Moreover, using the output of the observer, LQT was able to force the measured state to follow the reference and to stabilise it. In order to slow down the speed of degradation, the value of the matrix Q_1 can be increased so that the controller prioritize reducing the rate of evolution of degradation over the performance of the system, thus preventing the system's breakdown.

The convergence's proof of the gains K_k^x, K_k^d and K_k^q, as well as the matrices P_{k+1} and S_{k+1} needs to be investigated, therefore future work will focus on proving the convergence of the controller gains. Furthermore, the integration of remaining useful life in the cost function to extend its value will also be examined.

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