

# Perturbation Analysis of Subspace-Based Methods in Estimating a Damped Complex Exponential

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**Abstract**—We present a study of mode variance statistics for three SVD-based estimation methods in the case of a single-mode damped exponential. The methods considered are namely Kumaresan-Tufts, matrix pencil and Kung’s direct data approximation. Through first-order perturbation analysis, we derive closed-form expressions of the variance of the complex mode, frequency and damping factor estimates. These expressions are used to compare the different methods and to determine the optimal prediction order for matrix pencil and direct data approximation methods. Application to the undamped case shows the coherence of the results with those already stated in the literature. It is also found that the variances converge linearly towards the Cramér-Rao bound. Finally, the theoretical results are verified using Monte Carlo simulations.

**Index Terms**—Damped exponential model, direct data approximation, linear prediction, matrix pencil, perturbation analysis.

## I. INTRODUCTION

THE question of estimating model parameters of exponential signals in noise is a fundamental problem in signal processing. It has applications in several areas, including array processing, radar scattering, and nuclear magnetic resonance spectroscopy. In this context, several algorithms have been developed, including maximum likelihood approaches [1], [2] and subspace-based methods such as MUSIC [3], [4], backward linear prediction (BLP) [5], state-space [6], ESPRIT [7], and matrix pencil (MP) [8]. Statistical performances of these methods, at high signal-to-noise ratio (SNR), have also been extensively studied in the case of pure sinusoids [4], [9]–[14] and damped ones [15]–[18]. Most of these analyses are based on perturbation theory. For instance, Okhovat *et al.* [16] have studied BLP and direct data approximation (DDA) [6] methods, in the case of a single damped mode. The achieved expressions of variance come in the form of multiple sums, which is not very convenient. In [17], the authors consider the multimodal damped case using BLP. The resulting matrix expression is compact but does not give much insight about the actual performances. Finally, in [8], the MP method is studied in the multiple mode case. Here again, the variances come in the form of matrix expressions. However, the performances of the method have been clearly stated as closed-form expressions in the case of a single undamped exponential.

In a manner similar to [16], [17], the present work uses Wilkinson’s approach [19] to derive the expressions of the mode variance. The three methods discussed previously are studied in the case of a single noisy damped complex exponential. The first technique considered is the popular Kumaresan-Tufts method [5]. It performs a reduced rank pseudoinverse of a data matrix to get backward linear prediction coefficients, from which the signal modes are obtained through polynomial rooting. The matrix pencil method, introduced by Hua and Sarkar [8], is based on a matrix prediction equation in which the data matrices have a Hankel structure similar to that found in BLP.

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The last method considered here is Kung’s state-space direct data approximation method [6]. As will be seen in Section II, subspace-based methods that operate directly on data share a common step which amounts to find a reduced rank pseudoinverse of a data matrix. So the three aforementioned methods are studied in Section III, starting from the first-order perturbation analysis of the singular values and vectors, assuming a high SNR. Then, in Section IV, it is shown that these estimators are unbiased. Furthermore, closed-form expressions of the variance of the complex mode and the corresponding frequency and damping factor are derived. This enables us to establish the expression of the optimal tuning parameter of MP and DDA. In order to check the consistency of our results with those stated in the literature, the known equivalence between MP and DDA, for a first-order approximation, is shown again using the approach chosen. In the same manner, the frequency variance expression for an undamped exponential is given. In Section V, we demonstrate the superiority of MP and DDA over BLP in the single damped/undamped mode case, and we prove the convergence of the variances towards the Cramér-Rao bound. Finally, in Section VI, simulation results are presented to verify the theoretical expressions.

## II. ESTIMATION METHODS

The noise-perturbed exponential signal model is given by

$$\tilde{x}(n) = x(n) + e(n) = \sum_{i=1}^M a_i p_i^n + e(n) \quad (1)$$

for  $n = 0, \dots, N - 1$ . Here  $p_i = \exp(\alpha_i + j\omega_i) = r_i \exp(j\omega_i)$ ,  $i = 1, \dots, M$  are the signal modes ( $\alpha_i < 0$  and  $r_i < 1$ ) with complex amplitudes  $a_i = A_i \exp(j\phi_i)$ . The term  $e(n)$  is a zero-mean complex white Gaussian noise with variance  $\sigma_e^2$ ; so the real and imaginary parts of  $e(n)$  are assumed to be independent and of equal variances. Model (1) is used in this section to present the principles of the estimation techniques considered. Then, for perturbation analysis, we consider only the single-mode case, i.e.  $M = 1$ , and we use the following signal model instead of model (1):

$$\tilde{x}(n) = x(n) + e(n) = ap^n + e(n). \quad (2)$$

Throughout this paper, the notation  $\tilde{X}$  refers to the noisy or perturbed version of the quantity  $X$ , i.e.  $\tilde{X} = X + \Delta X$ , where  $X$  is a scalar or matrix and  $\Delta X$  the error term (or noise). Matrices are denoted by bold capital letters and vectors by bold lowercase letters.

### A. BLP Method

The BLP method [5] is based on backward linear prediction and uses a reduced rank approximation of the data matrix in order to decrease noise influence. It is made up of the following steps:

- 1) Using the available data, form the system of equations

$$\tilde{\mathbf{X}}_1 \tilde{\mathbf{b}} \approx -\tilde{\mathbf{x}}_0 \quad (3)$$

where  $\tilde{\mathbf{X}}_1 = [\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \dots, \tilde{\mathbf{x}}_L]$  and  $\tilde{\mathbf{x}}_k = [\tilde{x}(k), \tilde{x}(k+1), \dots, \tilde{x}(N-L-1+k)]^T$  for  $k = 0, \dots, L$ . The vector

$\tilde{\mathbf{b}}$  contains the prediction coefficients  $\{\tilde{b}_i\}_{i=1}^L$ , and  $L \geq M$  is the prediction order.

- 2) Perform the singular value decomposition (SVD) of matrix  $\tilde{\mathbf{X}}_1$  and set to 0 all but the first  $M$  largest singular values. The resulting matrix, denoted by  $\hat{\mathbf{X}}_1$ , is the best rank  $M$  approximation of  $\tilde{\mathbf{X}}_1$  in the Frobenius norm sense.
- 3) Compute the prediction vector estimate  $\tilde{\mathbf{b}}$  using the reduced rank pseudoinverse of  $\hat{\mathbf{X}}_1$ :

$$\tilde{\mathbf{b}} = -\hat{\mathbf{X}}_1^+ \tilde{\mathbf{x}}_0 \quad (4)$$

where the superscript “+” denotes the Moore-Penrose pseudoinverse.

- 4) Obtain the roots  $\{\tilde{z}_i\}_{i=1}^L$  of the polynomial  $\tilde{B}(z) = 1 + \sum_{i=1}^L \tilde{b}_i z^{-i} = \prod_{i=1}^L (1 - \tilde{z}_i z^{-1})$ , and then select those located outside the unit circle. They correspond to the inverse of signal modes (i.e.  $\tilde{p}_i = 1/\tilde{z}_i$  for  $i = 1, \dots, M$ ).

### B. Matrix Pencil Method

For damped sinusoids, Hua and Sarkar’s matrix pencil method [8] consists of the following steps:

- 1) Form two matrices  $\tilde{\mathbf{X}}_0$  and  $\tilde{\mathbf{X}}_1$ . The matrix  $\tilde{\mathbf{X}}_1$  is the same as before and  $\tilde{\mathbf{X}}_0$  is a shifted version of the latter:  $\tilde{\mathbf{X}}_0 = [\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_{L-1}]$ .
- 2) Compute the low rank approximation  $\hat{\mathbf{X}}_1$  of  $\tilde{\mathbf{X}}_1$  using, as before, the SVD.
- 3) As for BLP, compute the reduced rank pseudoinverse of  $\hat{\mathbf{X}}_1$  to obtain the matrix estimate  $\tilde{\mathbf{Z}}$ :

$$\tilde{\mathbf{Z}} = \hat{\mathbf{X}}_1^+ \tilde{\mathbf{X}}_0. \quad (5)$$

- 4) The estimates of the modes are the inverse of the  $M$  eigenvalues of  $\tilde{\mathbf{Z}}$  lying outside the unit circle.

### C. DDA Method

The DDA method by Kung *et al.* [6] is based on state-space formalism. The signal  $x(n)$  is seen as the free response of a linear system with transition matrix  $\mathbf{F}$  having eigenvalues which are the signal modes. So, the problem is to estimate the matrix  $\mathbf{F}$ , which can be done as follows:

- 1) Form the data matrix  $\tilde{\mathbf{X}} = [\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_L]$ .
- 2) Obtain its SVD which is partitioned as follows:

$$\tilde{\mathbf{X}} = \tilde{\mathbf{U}}' \tilde{\mathbf{S}}' \tilde{\mathbf{V}}'^H = \begin{bmatrix} \tilde{\mathbf{U}}'_1 & \tilde{\mathbf{U}}'_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{S}}'_1 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{S}}'_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{V}}_1'^H \\ \tilde{\mathbf{V}}_2'^H \end{bmatrix} \quad (6)$$

where  $\tilde{\mathbf{S}}'$  is a diagonal matrix containing the singular values of  $\tilde{\mathbf{X}}$  ordered in non-increasing fashion, and  $\tilde{\mathbf{S}}'_1$  is an  $M$ -by- $M$  diagonal matrix. The superscript “ $H$ ” denotes the conjugate transpose.

- 3) Estimate the observability matrix

$$\tilde{\Theta} = \tilde{\mathbf{U}}_1' \tilde{\mathbf{S}}_1'^{1/2}. \quad (7)$$

An estimate of  $\mathbf{F}$  is then given by

$$\tilde{\mathbf{F}} = \tilde{\Theta}_1^+ \tilde{\Theta}_2 \quad (8)$$

where  $\tilde{\Theta}_1$  (resp.  $\tilde{\Theta}_2$ ) is deduced from  $\tilde{\Theta}$  by eliminating the last (resp. the first) row.

- 4) The  $M$  eigenvalues of  $\tilde{\mathbf{F}}$  are the estimated modes.

## III. SINGULAR VALUE AND SINGULAR VECTOR PERTURBATIONS

All the methods presented before use the reduced rank pseudoinverse of data matrices ( $\tilde{\mathbf{X}}_1$  and  $\tilde{\mathbf{X}}$ ). So we start our study with the perturbation in the singular values and vectors of  $\tilde{\mathbf{X}}_1$  and  $\tilde{\mathbf{X}}$  for  $M = 1$ . We denote  $\tilde{\mathbf{X}}_1 = \mathbf{X}_1 + \mathbf{E}_1$ , where  $\mathbf{X}_1$  and  $\mathbf{E}_1$  are constructed as  $\tilde{\mathbf{X}}_1$  using  $x(n)$  and  $e(n)$  respectively. The notations  $\tilde{\mathbf{X}}_0 = \mathbf{X}_0 + \mathbf{E}_0$ ,  $\tilde{\mathbf{X}} = \mathbf{X} + \mathbf{E}$ , and  $\tilde{\mathbf{x}}_0 = \mathbf{x}_0 + \mathbf{e}_0$  are defined similarly.

### A. Matrix $\tilde{\mathbf{X}}_1$

It can be shown that the noiseless data matrix  $\mathbf{X}_1$  is rank 1 and its SVD is  $\mathbf{X}_1 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^H$ , where  $\sigma_1 = Ar\sqrt{k_v k_u}$ ,  $\mathbf{u}_1 = \frac{e^{j\phi}}{\sqrt{k_u}} [1, p, \dots, p^{N-L-1}]^T$ ,  $\mathbf{v}_1 = \frac{e^{-j\omega}}{\sqrt{k_v}} [1, p^*, \dots, p^{*L-1}]^T$ ,  $k_v = \sum_{i=0}^{L-1} r^{2i}$ , and  $k_u = \sum_{i=0}^{N-L-1} r^{2i}$ . The singular value  $\sigma_1$  is the square-root of the unique non-zero eigenvalue of the matrix  $\mathbf{X}_1^H \mathbf{X}_1$ , associated to the eigenvector  $\mathbf{v}_1$ . In the noisy case, we have

$$\tilde{\mathbf{X}}_1^H \tilde{\mathbf{X}}_1 = \mathbf{X}_1^H \mathbf{X}_1 + (\mathbf{E}_1^H \mathbf{E}_1 + \mathbf{E}_1^H \mathbf{X}_1 + \mathbf{X}_1^H \mathbf{E}_1). \quad (9)$$

At high SNR, the first-order perturbation of the eigenvalue  $\sigma_1^2$  of  $\mathbf{X}_1^H \mathbf{X}_1$  is given by [19]:

$$\Delta \sigma_1^2 = \mathbf{v}_1^H (\mathbf{E}_1^H \mathbf{X}_1 + \mathbf{X}_1^H \mathbf{E}_1) \mathbf{v}_1 = \sigma_1 (\mathbf{v}_1^H \mathbf{E}_1^H \mathbf{u}_1 + \mathbf{u}_1^H \mathbf{E}_1 \mathbf{v}_1). \quad (10)$$

### B. Matrix $\tilde{\mathbf{X}}$

The singular value decomposition of  $\mathbf{X}$  is given by  $\mathbf{X} = \sigma'_1 \mathbf{u}'_1 \mathbf{v}'_1{}^H$ , where  $\sigma'_1 = Ar\sqrt{k_v' k_u'}$ ,  $\mathbf{u}'_1 = \frac{e^{j\phi}}{\sqrt{k_u'}} [1, p, \dots, p^L]^T$ ,  $\mathbf{v}'_1 = \frac{e^{-j\omega}}{\sqrt{k_v'}} [1, p^*, \dots, p^{*N-L-1}]^T$ ,  $k_v' = \sum_{i=0}^{N-L-1} r^{2i}$ , and  $k_u' = \sum_{i=0}^L r^{2i}$ . In the case of DDA, we also need the expression of the first-order perturbation of the singular vector  $\mathbf{u}'_1$ . Let  $\tilde{\mathbf{u}}'_1 = \mathbf{u}'_1 + \Delta \mathbf{u}'_1$ , then [19]:

$$\Delta \mathbf{u}'_1 = \frac{1}{\sigma'_1} \sum_{i=2}^{L+1} (\mathbf{u}_i'^H \mathbf{E} \mathbf{v}'_1) \mathbf{u}'_i = \frac{1}{\sigma'_1} \sum_{i=2}^{L+1} \gamma_i \mathbf{u}'_i. \quad (11)$$

where  $\{\mathbf{u}'_i\}_{i=1}^{L+1}$  is the set of left-singular vectors of the data matrix  $\mathbf{X}$ , whose noisy counterparts are given by the columns of matrix  $\tilde{\mathbf{U}}'$  in (6).

## IV. ANALYSIS OF THE ESTIMATION METHODS

### A. BLP Method

At high SNR, matrix  $\tilde{\mathbf{X}}_1$  is approximately rank 1, thus [16]

$$\tilde{\mathbf{X}}_1^+ \approx \frac{\tilde{\mathbf{X}}_1^H}{\tilde{\sigma}_1^2}. \quad (12)$$

From (4), we can deduce that

$$\tilde{\mathbf{b}} \approx -\frac{1}{\tilde{\sigma}_1^2} \tilde{\mathbf{X}}_1^H \tilde{\mathbf{x}}_0. \quad (13)$$

Since  $\tilde{\mathbf{b}} = \mathbf{b} + \Delta \mathbf{b}$ , and  $\mathbf{b} = \frac{-1}{r\sqrt{k_v}} \mathbf{v}_1$ , the first-order perturbation of  $\mathbf{b}$  is

$$\Delta \mathbf{b} = -\frac{1}{\sigma_1^2} (\mathbf{X}_1^H \mathbf{e}_0 + \mathbf{E}_1^H \mathbf{x}_0 + \mathbf{b} \Delta \sigma_1^2). \quad (14)$$

The error  $\Delta \mathbf{b}$  in the prediction coefficients induces a shifting of the root  $z_1 = 1/p$  of the polynomial  $B(z)$  towards a new position  $\tilde{z}_1 = z_1 + \Delta z_1$ , where [20]:

$$\Delta z_1 = -\sum_{k=1}^L \frac{z_1^{L-k}}{\prod_{i=2}^L (z_1 - z_i)} \Delta b_k \quad (15)$$

and  $\{z_i\}_{i=2}^L$  are the zeros of the polynomial  $B(z)$  which are different from  $z_1$ . Let  $\beta_1 = \prod_{i=2}^L (z_1 - z_i) = \frac{1}{p^{L-1}} \frac{k_v - Lr^{2L}}{1-r^{2L}}$  and  $\mathbf{g}_1 = [z_1^{L-1}, z_1^{L-2}, \dots, 1]^H = \frac{r\sqrt{k_v}}{p^{*L}} \mathbf{v}_1$ , then (15) may be rewritten as

$$\Delta z_1 = -\frac{1}{\beta_1} \mathbf{g}_1^H \Delta \mathbf{b} = -\frac{r\sqrt{k_v}}{\beta_1 p^L} \mathbf{v}_1^H \Delta \mathbf{b}. \quad (16)$$

After some straightforward calculations using (16) and (14), we finally obtain:

$$\Delta z_1 = \frac{1}{\sigma_1 \beta_1 p^L} \mathbf{u}_1^H (r\sqrt{k_v} \mathbf{e}_0 - \mathbf{E}_1 \mathbf{v}_1). \quad (17)$$

It can be seen that the estimate  $\tilde{z}_1$  is unbiased since  $\mathbb{E}\{\Delta z_1\} = 0$ . Since  $\tilde{z}_1 = 1/\tilde{p} = (\tilde{\alpha} + j\tilde{\omega})^{-1}$ , we obtain using a first-order series expansion  $\Delta z_1 \approx -(\Delta\alpha + j\Delta\omega)/p$ . Hence the estimates  $\tilde{\alpha}$  and  $\tilde{\omega}$  are also unbiased:  $\mathbb{E}\{\Delta\alpha\} = \mathbb{E}\{\Delta\omega\} = 0$ . Now, using the fact that  $e(n)$  is zero-mean uncorrelated complex noise and after some lengthy calculations, it can be shown that  $\mathbb{E}\{(\Delta z_1)^2\} = 0$ , and

$$\mathbb{E}\{|\Delta z_1|^2\} = \frac{A^2 \sigma_e^2}{\sigma_1^4 r^2} \left( \frac{1-r^{2L}}{k_v - Lr^{2L}} \right)^2 (r^4 k_v^2 k_u + s_2 - 2r^2 k_v s_1) \quad (18)$$

with  $s_1 = \sum_{i=0}^{m-1} ir^{2i} + m \sum_{i=m}^{N-L-1} r^{2i}$ ,  $s_2 = \sum_{i=0}^{m-1} i^2 r^{2i} + m^2 \sum_{i=m}^{N-m-1} r^{2i} + \sum_{i=N-m}^{N-1} (N-i)^2 r^{2i}$ , and  $m = \min\{L, N-L\}$ . Moreover,  $\text{var}(\Delta\alpha) = \text{var}(\Delta\omega) = \frac{\sigma_e^2}{2} \mathbb{E}\{|\Delta z_1|^2\}$ , which implies:

$$\text{var}(\Delta\omega) = \frac{\sigma_e^2}{2A^2 k_v^2 k_u^2 r^4} \left( \frac{1-r^{2L}}{k_v - Lr^{2L}} \right)^2 (r^4 k_v^2 k_u + s_2 - 2r^2 k_v s_1). \quad (19)$$

Finally, replacing all the sums leads to expression (23) given at the bottom of the page. This new result is interesting since it is easily exploitable. Namely, it may be used to compare the performances of BLP to MP and DDA under the assumption of high SNR. This will be done in Section V. For the particular case of an undamped sinusoid ( $r = 1$ ), the frequency variance reduces to:  $\text{var}(\Delta\omega) = \frac{\sigma_e^2}{A^2} \frac{2(2L+1)}{3L(L+1)(N-L)^2}$  if  $L \leq N/2$  and  $\text{var}(\Delta\omega) = \frac{\sigma_e^2}{A^2} \frac{2[-(N-L)^2 + 3L^2 + 3L + 1]}{3L^2(L+1)^2(N-L)}$  if  $L \geq N/2$ , which is consistent with the results in [8], [14].

### B. Matrix Pencil Method

Starting from (5) and (12), and using the fact that  $\tilde{\mathbf{X}}_0 = \mathbf{X}_0 + \mathbf{E}_0$ , we obtain the following expression:

$$\Delta \mathbf{Z} = \tilde{\mathbf{Z}} - \mathbf{Z} \approx \frac{1}{\sigma_1^2} (-\mathbf{Z} \Delta \sigma_1^2 + \mathbf{X}_1^H \mathbf{E}_0 + \mathbf{E}_1^H \mathbf{X}_0). \quad (20)$$

The first-order perturbation of the eigenvalue  $z_1$  of the matrix  $\mathbf{Z}$  is then [19]

$$\Delta z_1 = \frac{1}{\sigma_1^2} \mathbf{v}_1^H (-\mathbf{Z} \Delta \sigma_1^2 + \mathbf{X}_1^H \mathbf{E}_0 + \mathbf{E}_1^H \mathbf{X}_0) \mathbf{v}_1. \quad (21)$$

As  $\mathbf{Z} = \frac{1}{p} \mathbf{v}_1 \mathbf{v}_1^H$ , replacing  $\Delta \sigma_1^2$  by its expression in (10), we get

$$\Delta z_1 = \frac{1}{\sigma_1 p} \mathbf{u}_1^H (p \mathbf{E}_0 - \mathbf{E}_1) \mathbf{v}_1. \quad (22)$$

Applying mathematical expectation, it leads to  $\mathbb{E}\{\Delta z_1\} = 0$ , which implies that the estimates  $\tilde{\alpha}$  and  $\tilde{\omega}$  are unbiased:  $\mathbb{E}\{\Delta\alpha\} =$

$\mathbb{E}\{\Delta\omega\} = 0$ . As for BLP, it can also be shown that  $\mathbb{E}\{(\Delta z_1)^2\} = 0$ , and

$$\mathbb{E}\{|\Delta z_1|^2\} = \frac{A^2 \sigma_e^2}{\sigma_1^4 r^2} ((1+r^2)s_2 - 2r^2 s_2') \quad (24)$$

where  $s_2' = s_2 + \sum_{i=0}^{m-1} ir^{2i} - \sum_{i=N-m}^{N-1} (N-i)r^{2i}$ . Thus  $\text{var}(\Delta\alpha) = \text{var}(\Delta\omega)$ , with:

$$\text{var}(\Delta\omega) = \frac{\sigma_e^2}{2A^2 k_v^2 k_u^2 r^4} ((1+r^2)s_2 - 2r^2 s_2'). \quad (25)$$

Again, replacing all the sums leads to:

$$\text{var}(\Delta\omega) = \frac{\sigma_e^2 (1-r^2)^3}{2A^2 r^2} \times \begin{cases} \frac{1+r^{2N-2L}}{(1-r^{2N-2L})^2 (1-r^{2L})} & \text{if } L \leq N/2 \\ \frac{1+r^{2L}}{(1-r^{2N-2L})(1-r^{2L})^2} & \text{if } L \geq N/2 \end{cases} \quad (26)$$

We observe that  $\text{var}(\Delta\omega)$  is a rational function in  $r^{2L}$ , so it is possible here to obtain the optimal value of  $L$  for which the variance is minimized. For instance, for  $L \leq N/2$ , the first derivative of  $\text{var}(\Delta\omega)$  with respect to  $t = r^{2L}$  is zero when:

$$t^3 + 3r^{2N} t^2 - 3r^{2N} t - r^{4N} = 0. \quad (27)$$

This is a cubic equation in  $t$  which may be solved analytically using, for example, Cardano's formula [21]. The value of  $t$  is found to be  $t = r^N / \tan((\pi - \tan^{-1} r^{-N})/3)$ , which implies

$$L_{\min} = \frac{N}{2} - \frac{1}{2 \ln r} \ln \left( \tan \frac{\pi - \tan^{-1} r^{-N}}{3} \right). \quad (28)$$

One special case of interest is when  $r = 1$ , for which we obtain the well-known optimal value for an undamped sinusoid:  $L_{\min} = N/3$ . On the contrary, when  $r$  tends towards zero, we are confronted with a damped wave with a strong damping. In the latter case,  $L_{\min}$  tends towards  $N/2$ . This value is also reached asymptotically (as  $N \rightarrow \infty$ ), for any  $r < 1$ . So the optimal value of  $L$  lies between  $N/3$  and  $N/2$  and approaches  $N/2$  as the damping factor increases. Of course, since  $\text{var}(\Delta\omega)$  is symmetric about  $L = N/2$  and assuming  $N$  even, the variance reaches the same minimum at  $N - L_{\min}$ .

Finally, the variance in the undamped case may be derived easily from (26), and corresponds to the one presented in [8]:  $\text{var}(\Delta\omega) = \frac{\sigma_e^2}{A^2} \frac{1}{(N-L)^2 L}$  if  $L \leq N/2$  and  $\text{var}(\Delta\omega) = \frac{\sigma_e^2}{A^2} \frac{1}{(N-L)L^2}$  if  $L \geq N/2$ .

### C. DDA Method

In the single mode case, the matrices  $\tilde{\Theta}_1$  and  $\tilde{\Theta}_2$  in (8) are vectors. The pseudoinverse of  $\tilde{\Theta}_1$  is then  $\tilde{\Theta}_1^+ = \frac{1}{\tilde{\kappa}} \tilde{\Theta}_1^H$ , in which  $\tilde{\kappa} = \|\tilde{\Theta}_1\|^2$ . Let  $\tilde{\kappa} = \kappa + \Delta\kappa$ , the perturbation of  $\kappa$  is then

$$\Delta\kappa = \Theta_1^H \Delta\Theta_1 + \Delta\Theta_1^H \Theta_1 \quad (29)$$

from which we get

$$\tilde{\Theta}_1^+ \approx \Theta_1^+ + \frac{1}{\kappa} (\Delta\Theta_1^H - \frac{1}{\kappa} \Theta_1^H \Delta\kappa). \quad (30)$$

Then, using (8) and according to (29), the following expression of the perturbation of  $\mathbf{F}$  (which in our case equals the scalar  $p$ ) can be derived:

$$\Delta p = \frac{1}{\kappa} \Theta_1^H (\Delta\Theta_2 - p \Delta\Theta_1). \quad (31)$$

Since  $\tilde{S}_1$  in (7) is a scalar, it simplifies itself in (8). Consequently, one may simply choose  $\tilde{\Theta} = \tilde{\mathbf{u}}_1^H$  (and  $\Theta = \mathbf{u}_1^H$ ), which leads

$$\text{var}(\Delta\omega) = \frac{\sigma_e^2 (1-r^2)^3}{2A^2 r^2} \times \begin{cases} \frac{(1+r^{2N-2L})[1-(2L+1)(1-r^2)r^{2L}-r^{4L+2}]}{(1-r^{2N-2L})^2 [1-r^{2L}-L(1-r^2)r^{2L}]^2} & \text{if } L \leq N/2 \\ \frac{(1-r^{2N-2L})(1+r^{2L})(1+r^{2L+2})-2(N-L)(1-r^2)(1+r^{2N-2L})r^{2L}}{(1-r^{2N-2L})^2 [1-r^{2L}-L(1-r^2)r^{2L}]^2} & \text{if } L \geq N/2 \end{cases} \quad (23)$$

to  $\mathbb{E}\{\Delta p\} = 0$  in view of (11). Therefore, the estimates of the frequency and the damping factor are also unbiased. After some simplifications, we get  $\mathbb{E}\{(\Delta p)^2\} = 0$ , and

$$\mathbb{E}\{|\Delta p|^2\} = \frac{\sigma_e^2}{\kappa^2 A^2 k_v^2 k_u^2 r^2} ((1+r^2)s_2 - 2r^2 s_2'). \quad (32)$$

Finally, using the fact that  $\text{var}(\Delta\alpha) = \text{var}(\Delta\omega) = \frac{1}{2r^2} \mathbb{E}\{|\Delta p|^2\}$ ,  $\kappa = k_v/k_u$  and  $k_{v'} = k_u$ , we obtain:

$$\text{var}(\Delta\omega) = \frac{\sigma_e^2}{2A^2 k_v^2 k_u^2 r^4} ((1+r^2)s_2 - 2r^2 s_2'). \quad (33)$$

From to (33) and (25), it appears that DDA and MP are equivalent for a first-order approximation. Of course, this result had already been established in [22] for the general case of  $M \geq 1$  exponentials, using a different approach. So all the properties demonstrated before on MP are also valid for DDA.

## V. PERFORMANCE COMPARISON AND CONVERGENCE TO THE CRAMÉR-RAO BOUND

Here we demonstrate the superiority of MP and DDA over BLP, for a first-order approximation. Then, we discuss the asymptotic convergence of all the variances towards the Cramér-Rao lower Bound (CRB).

**Proposition 1** For all  $r \in (0, 1]$  and  $1 \leq L \leq N-1$ , the frequency variance achieved by BLP is at least equal to that obtained by MP and DDA for a first-order approximation, that is:

$$\text{var}(\Delta\omega)^{BLP} \geq \text{var}(\Delta\omega)^{MP} = \text{var}(\Delta\omega)^{DDA}. \quad (34)$$

*Proof:* See Appendix A. ■

This result was already established in the case of a single pure sinusoid (see e.g. [8]) and is now proven in the damped case.

For any unbiased estimator of the frequency and the damping factor, under the assumption of a single exponential model, the CRB is given by [23]:

$$CRB(\alpha) = CRB(\omega) = \frac{\sigma_e^2}{2A^2} \frac{(1-r^2)^3(1-r^{2N})}{r^2(1-r^{2N})^2 - N^2 r^{2N}(1-r^2)^2}. \quad (35)$$

As  $N \rightarrow \infty$ , the convergence of all the previous estimation variances towards the CRB is obvious in view of (23), (26) and (35), for any given value of  $L$  such that  $L = \mu N$ ,  $\mu \in (0, 1)$ , provided that  $r < 1$ . The following proposition gives the rate of convergence.

**Proposition 2** Let  $\varepsilon_N$  be the deviation of one of the variances in (23), (26) or (33) from the CRB:  $\varepsilon_N = \text{var}(\Delta\omega) - CRB(\omega)$ . Then,  $\forall r \in (0, 1)$  and  $L = \mu N$ ,  $\mu \in (0, 1)$ ,  $\varepsilon_N$  converges linearly towards zero with the following rates:

$$\lim_{N \rightarrow \infty} \frac{\varepsilon_{N+1}}{\varepsilon_N} = \begin{cases} r^{2\mu} & \text{if } \mu \in (0, 1/2] \\ r^{2(1-\mu)} & \text{if } \mu \in [1/2, 1). \end{cases} \quad (36)$$

*Proof:* See Appendix B. ■

Proposition 2 gives the rate of convergence to the CRB if  $L$  is chosen such that  $L = \mu N$ . For instance, the rate of convergence is  $r$  for  $L = N/2$ , and  $r^{2/3}$  for  $L \in \{N/3, 2N/3\}$ . In the same manner, it can be shown that, for the particular case of MP and DDA, the minimum variance corresponding to  $L_{\min}$  in (28) converges to the CRB with rate  $r$ .

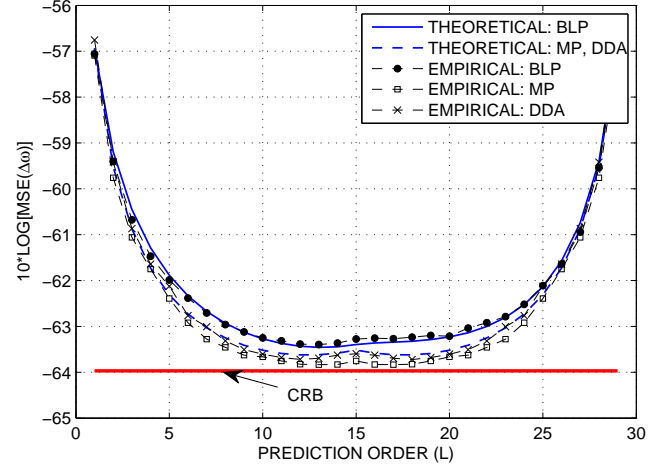


Fig. 1. Theoretical and empirical MSEs for BLP, MP and DDA versus prediction order (SNR = 40 dB).

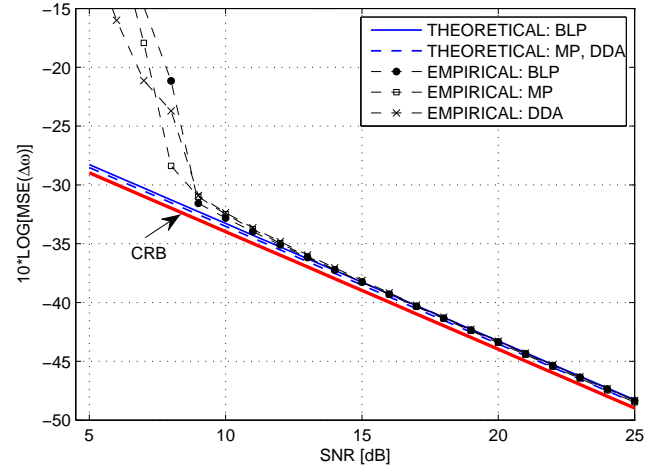


Fig. 2. Theoretical and empirical MSEs for BLP, MP and DDA versus SNR ( $L = 10$ ).

## VI. NUMERICAL SIMULATIONS

We consider a signal composed of one damped exponential with parameters  $\alpha = -0.1$ ,  $\omega = 0.2$ , and  $N = 30$ . The peak SNR is fixed to 40 dB:  $10 \log(A^2/\sigma_e^2) = 40$ . Fig. 1 shows the theoretical and sample mean square errors (MSEs) obtained from 1,000 realizations of additive noise for BLP, MP and DDA, together with the CRB. It can be seen that the theoretical MSEs are close to the estimated ones. Moreover, it appears clearly, as demonstrated analytically before, that MP and DDA perform better than BLP. Note that this result is valid whatever the value of the damping, assuming a sufficiently high SNR. We also observe that the minimum MSE for MP and DDA is attained at  $L = 12$  and  $L = 18$ , which correspond to the values obtained from expression (28):  $L_{\min} = 12.44$  and  $N - L_{\min} = 17.66$ .

For the second example, the same signal is used but the SNR is now varying. The prediction order is set to  $L = 10$ . The results achieved are given in Fig. 2. Here we observe that the theoretical expressions of the variance are valid beyond a threshold SNR, which in this case is about 8 dB. Of course, this is not a rule of thumb because, in fact, it also depends on the damping factor.

## VII. CONCLUSION

In this paper, we have presented a first-order perturbation analysis of three subspace-based techniques operating directly on the data: Kumaresan-Tufts, matrix pencil and direct data approximation, in the case of a single damped exponential. We have derived the analytical closed-form expressions of the mode, frequency and damping factor variances. Thanks to these expressions, we have shown that MP and DDA perform better than BLP. Moreover, we have found the optimal prediction order for MP and DDA. In fact, this order depends not only on the number of samples but also on the damping, which is unknown. So, in practice, an appropriate value will lie between  $N/3$  and  $N/2$ , the latter being preferable for a strongly damped sinusoid. Note that, unless some restrictive hypotheses on the signal parameters are stated, the expression of the optimal prediction order cannot be found for BLP, due to the nonlinear aspect of the underlying problem. The extension of the results to the multimodal case is possible provided that the modes are separated enough and the damping factors are of the same order of magnitude.

## APPENDIX A

## PROOF OF PROPOSITION 1

Here we prove (34) for  $r \in (0, 1)$ . The case  $r = 1$  is simple and will not be considered. For  $L \leq N/2$ , the ratio of BLP and MP variances is

$$\frac{\text{var}(\Delta\omega)^{BLP}}{\text{var}(\Delta\omega)^{MP}} - 1 = r^{2L} \frac{r^2(1-r^{2L})^2 - L^2(1-r^2)^2 r^{2L}}{[1-r^{2L} - L(1-r^2)r^{2L}]^2}. \quad (37)$$

Since  $\sum_{i=0}^{L-1} r^{2i} = (1-r^{2L})/(1-r^2)$  and  $\prod_{i=0}^{L-1} r^{2i} = r^{L(L-1)}$ , using the arithmetic mean–geometric mean inequality, we obtain  $(1-r^{2L}) \geq L(1-r^2)r^{L-1}$ . Thus  $r^2(1-r^{2L})^2 \geq L^2(1-r^2)^2 r^{2L}$ .

For  $L \geq N/2$ , the ratio of BLP and MP variances is

$$\frac{\text{var}(\Delta\omega)^{BLP}}{\text{var}(\Delta\omega)^{MP}} - 1 = \frac{r^{2L}(1+r^{2L})[r^2(1-r^{2L})^2 + 2L(1-r^2)(1-r^{2L}) - L^2(1-r^2)^2 r^{2L}]}{(1+r^{2L})[1-r^{2L} - L(1-r^2)r^{2L}]^2} - \frac{2r^{2L}(N-L)(1-r^2)(1+r^{2N-2L})(1-r^{2L})^2}{(1-r^{2N-2L})(1+r^{2L})[1-r^{2L} - L(1-r^2)r^{2L}]^2}. \quad (38)$$

Using, once again, the arithmetic mean–geometric mean inequality on the sequence  $\{r^{2n}\}_{n=0}^{L-1}$ , it yields  $L^2(1-r^2)^2 r^{2L} \leq r^{2L}(1-r^{2L})^2$ , so

$$\frac{\text{var}(\Delta\omega)^{BLP}}{\text{var}(\Delta\omega)^{MP}} - 1 \geq \frac{2r^{2L}(1-r^2)(1-r^{2L})^2}{(1+r^{2L})[1-r^{2L} - L(1-r^2)r^{2L}]^2} \times \left[ \frac{L(1+r^{2L})}{1-r^{2L}} - \frac{(N-L)(1+r^{2N-2L})}{1-r^{2N-2L}} \right]. \quad (39)$$

It can be shown that the sequence  $w_n(r) = n(1+r^{2n})/(1-r^{2n})$ ,  $n \geq 1$ , increases for all  $r \in (0, 1)$ . So, using the fact that  $L \geq N-L$ , we obtain  $w_L(r) \geq w_{N-L}(r)$ , which completes the proof of Proposition 1.

## APPENDIX B

## PROOF OF PROPOSITION 2

In this appendix, we demonstrate the linear convergence of the variance towards the CRB for the MP method with parameter  $L = \mu N$  such that  $0 < \mu \leq 1/2$ . For  $L \leq N/2$ , we have:

$$\varepsilon_N = \frac{\sigma_e^2(1-r^2)^3}{2A^2 r^2} \times \frac{W_N}{(1-r^{2(1-\mu)N})^2(1-r^{2\mu N})[r^2(1-r^{2N})^2 - N^2 r^{2N}(1-r^2)^2]} \quad (40)$$

where  $\varepsilon_N = \text{var}(\Delta\omega) - \text{CRB}(\omega)$ , and

$$W_N = r^{2\mu N} \left\{ r^2(1-r^{2N})[1-r^{2(1-2\mu)N}(3+3r^{2\mu N}+r^{2(1-\mu)N})] - N^2 r^{2(1-\mu)N}(1+r^{2(1-\mu)N}) \right\}. \quad (41)$$

Since  $\mu \leq 1/2$ , it is easy to see that  $\varepsilon_N \rightarrow 0$  as  $N \rightarrow \infty, \forall r \in (0, 1)$ . The rate of convergence is then

$$\rho = \lim_{N \rightarrow \infty} \frac{\varepsilon_{N+1}}{\varepsilon_N} = \lim_{N \rightarrow \infty} \frac{W_{N+1}}{W_N} = \lim_{N \rightarrow \infty} \frac{r^{2\mu(N+1)}}{r^{2\mu N}} = r^{2\mu}. \quad (42)$$

This shows that the convergence is linear since  $\rho \in (0, 1)$ . The cases of BLP and  $\mu > 1/2$  may be proved similarly.

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